

DERIVED CATEGORIES – A TEXTBOOK

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ABSTRACT. This is the first part of a book on *derived categories*. The purpose of the book is to provide solid foundations for the theory of derived categories, and to present several applications of this theory to *algebra* and *algebraic geometry*. The emphasis is on constructions and examples, rather than on axiomatics.

Here is a brief description of the book. After a review of standard facts on abelian categories, we study *differential graded algebra* in depth (including DG rings, DG modules, DG categories and DG functors).

We then move to *pretriangulated categories* and *triangulated functors* between them, and explain how they arise from the DG background. Specifically, we talk about the homotopy category $\mathbf{K}(A, \mathbf{M})$ of DG A -modules in \mathbf{M} , where A is a DG ring and \mathbf{M} is an abelian category.

The next topic is *localization of categories*, that we present in a very detailed way. We arrive at the derived category $\mathbf{D}(A, \mathbf{M})$, which is the localization of $\mathbf{K}(A, \mathbf{M})$ with respect to the quasi-isomorphisms. We define the left and right *derived functors* of a triangulated functor. Here we introduce *2-categorical language*, as an effective means to state the universal properties of the derived functors.

Two sections are devoted to *resolutions of DG modules*. We give a detailed construction of K-injective and K-projective resolutions in several important algebraic situations (including unbounded complexes of modules over rings).

The second part of the book is yet to be written, and below are the topics that are expected to be included in it. We will talk about *derived bifunctors*, with emphasis on the bifunctors $\mathbf{R}\mathrm{Hom}$ and \otimes^L . After that we shall study *derived categories in geometry*. Among the results are existence of K-injective and K-flat resolutions of unbounded complexes of sheaves on ringed spaces.

The rest of the book will cover more specialized topics. There will be a section about *derived categories in commutative algebra*. This will include the first textbook treatment of *rigid dualizing complexes*. Building on that, we will give a detailed exposition of *residues and duality in algebraic geometry*.

The final section will deal with *derived categories in noncommutative algebra*. We will talk about noncommutative dualizing and tilting complexes, and their interaction. We plan to finish the book with *derived Morita theory*, both in algebra and in algebraic geometry.

CONTENTS

First Part [written]	7
0. Introduction	7
0.1. A Motivating Discussion: Duality	7
0.2. Synopsis of the Book	12
0.3. Recommended Bibliography	14
0.4. Acknowledgments	15
1. Basic Facts on Categories	17
1.1. Set Theory	17
1.2. Notation	17
1.3. Epimorphisms and Monomorphisms	18
1.4. Products and Coproducts	19
1.5. Equivalence of Categories	19
1.6. Bifunctors	20
1.7. Representable Functors	20
2. Abelian Categories and Additive Functors	23
2.1. Linear Categories	23
2.2. Additive Categories	24
2.3. Abelian Categories	25
2.4. Additive Functors	28
2.5. Projective Objects	31
2.6. Injective Objects	32
3. Differential Graded Algebra	37
3.1. Graded Algebra	37
3.2. DG \mathbb{K} -modules	42
3.3. DG Rings and Modules	43
3.4. DG Categories	45
3.5. DG Functors	47
3.6. Complexes in Abelian Categories	48
3.7. The DG Category $\mathbf{C}(A, M)$	50
3.8. The Translation Functor	52
3.9. The Standard Cone of a Strict Morphism	55
4. Properties of DG Functors	57
4.1. The Gauge of a Graded Functor	57
4.2. The Translation Isomorphism of a DG Functor	58
4.3. Cones and DG Functors	59
4.4. Examples of DG Functors	62
5. Pretriangulated Categories and Triangulated Functors	67
5.1. T-Additive Categories	67
5.2. Pretriangulated Categories	68
5.3. Triangulated and Cohomological Functors	70
5.4. The Homotopy Category is Pretriangulated	73
6. Localization of Categories	81
6.1. The Formalism of Localization	81
6.2. Ore Localization	82
6.3. Localization of Linear Categories	92
6.4. Localization of Pretriangulated Categories	94

7. The Derived Category $\mathbf{D}(A, \mathbf{M})$	99
7.1. Definition of the Derived Category	99
7.2. Localization of Subcategories of $\mathbf{K}(A, \mathbf{M})$	100
7.3. Boundedness Conditions	102
7.4. Thick Subcategories of \mathbf{M}	104
7.5. The Embedding of \mathbf{M} in $\mathbf{D}(\mathbf{M})$	105
8. Derived Functors	107
8.1. 2-Categorical Notation	108
8.2. Some Preliminaries on Triangulated Functors	110
8.3. Right Derived Functors	111
8.4. Left Derived Functors	118
9. Resolutions of DG Modules	121
9.1. K-Injective DG Modules	121
9.2. K-Projective DG Modules	125
9.3. K-Flat DG Modules	127
10. Existence of Resolutions	129
10.1. Direct and Inverse Limits of Complexes	129
10.2. K-Projective Resolutions in $\mathbf{C}^-(\mathbf{M})$	133
10.3. K-Projective Resolutions in $\mathbf{C}(A)$	137
10.4. K-Injective Resolutions in $\mathbf{C}^+(\mathbf{M})$	143
10.5. K-Injective Resolutions in $\mathbf{C}(A)$	145
Second Part [to be written later]	155
11. Derived Bifunctors [later]	155
11.1. T-Additive Bifunctors [later]	155
11.2. Right Derived Bifunctors [later]	155
11.3. Left Derived Bifunctors [later]	155
11.4. The Right Derived Hom Bifunctor [later]	155
11.5. The Left Derived Tensor Bifunctor [later]	155
11.6. Adjunctions [later]	155
12. Derived Categories in Geometry [later]	155
12.1. Recalling Facts on Ringed Spaces	155
12.2. K-Flat Resolutions in $\mathbf{C}(\mathcal{A})$ [later]	155
12.3. K-Injective Resolutions in $\mathbf{C}(\mathcal{A})$ [later]	155
12.4. K-Flasque Resolutions in $\mathbf{C}(\mathcal{A})$ [later]	155
12.5. Standard Derived Functors in Geometry [later]	155
12.6. Survey: Poincaré-Verdier Duality [later, optional]	155
12.7. Survey: Applications to Birational Geometry [later, optional]	155
13. Derived Categories in Commutative Algebra [later]	155
13.1. Dualizing Complexes over Commutative Rings [later]	155
13.2. Local Duality [later]	155
13.3. Rigid Dualizing Complexes [later]	155
13.4. Survey: MGM Equivalence for Adic Rings [later, optional]	155
14. Residues and Duality in Algebraic Geometry [later]	155
14.1. Dualizing Complexes on Schemes [later]	155
14.2. Rigid Residue Complexes on Schemes [later]	155
14.3. Perverse Coherent Sheaves on Schemes [later, optional]	155
14.4. Grothendieck Duality for Proper Maps [later]	155

14.5. Survey of Related Material [later, optional]	155
15. Derived Categories in Noncommutative Algebra [later]	156
15.1. Noncommutative Dualizing Complexes [later]	156
15.2. Noncommutative Tilting Complexes [later]	156
15.3. The Noncommutative Derived Picard Group of a Ring [later]	156
15.4. Derived Morita Theory [later]	156
Back to First Part [written]	157
References	157

First Part [written]

0. INTRODUCTION

This book develops the theory of *derived categories*, starting from the foundations, and going all the way to applications in algebra and geometry. The emphasis is on explicit constructions (with examples), as opposed to axiomatics. The most abstract concept we use is probably that of abelian category (which seems indispensable).

A special feature of this book is that most of the theory deals with $\mathbf{D}(A, \mathbf{M})$, the *derived category of DG A -modules in \mathbf{M}* , where A is a DG (differential graded) ring and \mathbf{M} is an abelian category. This covers most important examples that arise in algebra and geometry:

- The derived category $\mathbf{D}(A)$ of DG A -modules, for any DG ring A . This includes ordinary rings.
- The derived category $\mathbf{D}(\mathbf{M})$ for any abelian category \mathbf{M} . This includes $\mathbf{M} = \mathbf{Mod} \mathcal{A}$, the category of sheaves of \mathcal{A} -modules on a ringed space (X, \mathcal{A}) .

Furthermore, we work with *unbounded* derived categories. We prove existence of resolutions (bounded or unbounded) in several contexts.

The first half of the book (Sections 1-10) covers the general theory. This is done in an unorthodox manner, using DG categories as the source of derived categories and triangulated functors. Another departure from the tradition is that we only consider *pretriangulated categories*, thus sparing ourselves the burden of the octahedral axiom. In this part of the book we provide detailed proofs of all statements (except the routine ones, that are left as exercises). A more detailed description of the contents of the first half is in the Synopsis (subsection 0.2 of the Introduction).

The second half of the book (that is not yet written) shall start off with more of the general theory: derived bifunctor, and derived categories in geometry. This is in Sections 11-12).

After that we shall deal with a few specialized topics:

- ▷ Derived Categories in Commutative Algebra.
- ▷ Residues and Duality in Algebraic Geometry.
- ▷ Derived Categories in Noncommutative Algebra.

In this last portion of the book we shall leave out some of the proofs (but there are precise external references). Much of the material here is the state of the art, and is not included in any prior textbook.

The book is based on notes for advanced courses given at Ben Gurion University, in the academic years 2011-12 and 2015-16. The main sources for the first part of the book are [RD] and [KaSc1]; but the DG theory component is absent from those earlier texts, and is pretty much our own interpretation of folklore results.

0.1. A Motivating Discussion: Duality. By way of introduction to the subject of derived categories, let us consider *duality*.

We begin with something elementary: linear algebra. Take a field \mathbb{K} . Given a \mathbb{K} -module M (i.e. a vector space), let

$$D(M) := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}),$$

be the dual module. There is a canonical homomorphism

$$\eta_M : M \rightarrow D(D(M)),$$

namely $\eta_M(m)(\phi) := \phi(m)$ for $m \in M$ and $\phi \in D(M)$. If M is finitely generated then η_M is an isomorphism (actually this is “if and only if”).

To formalize this situation, let $\mathbf{Mod} \mathbb{K}$ denote the category of \mathbb{K} -modules. Then

$$D : \mathbf{Mod} \mathbb{K} \rightarrow \mathbf{Mod} \mathbb{K}$$

is a contravariant functor, and

$$\eta : \mathrm{Id} \rightarrow D \circ D$$

is a natural transformation. Here Id is the identity functor of $\mathbf{Mod} \mathbb{K}$.

Now let us replace \mathbb{K} by any nonzero commutative ring A . Again we can define a contravariant functor

$$D : \mathbf{Mod} A \rightarrow \mathbf{Mod} A, \quad D(M) := \mathrm{Hom}_A(M, A),$$

and a natural transformation $\eta : \mathrm{Id} \rightarrow D \circ D$. It is easy to see that $\eta_M : M \rightarrow D(D(M))$ is an isomorphism if M is a finitely generated free module. Of course we can't expect reflexivity (i.e. η_M being an isomorphism) if M is not finitely generated; but what about a finitely generated module that is not free?

In order to understand this better, let us concentrate on the ring $A = \mathbb{Z}$. Since \mathbb{Z} -modules are just abelian groups, the category $\mathbf{Mod} \mathbb{Z}$ is often denoted by \mathbf{Ab} . Let \mathbf{Ab}_f be the full subcategory of finitely generated abelian groups. Any finitely generated abelian group is of the form $M \cong T \oplus F$, with F free and T finite. (The letters “T” and “F” stand for “torsion” and “free” respectively.) It is important to note that this is *not a canonical isomorphism*. There is a canonical short exact sequence

$$(0.1.1) \quad 0 \rightarrow T \xrightarrow{\phi} M \xrightarrow{\psi} F \rightarrow 0,$$

but the decomposition $M \cong T \oplus F$ comes from *choosing a splitting* $\sigma : F \rightarrow M$ of this sequence.

Exercise 0.1.2. Prove that the exact sequence (0.1.1) is functorial (i.e. natural); namely there are functors $T, F : \mathbf{Ab}_f \rightarrow \mathbf{Ab}_f$, and natural transformations $\phi : T \rightarrow \mathrm{Id}$ and $\psi : \mathrm{Id} \rightarrow F$, such that for any $M \in \mathbf{Ab}_f$, the group $T(M)$ is finite; the group $F(M)$ is free; and the sequence of homomorphisms

$$(0.1.3) \quad 0 \rightarrow T(M) \xrightarrow{\phi_M} M \xrightarrow{\psi_M} F(M) \rightarrow 0$$

is exact.

Next, prove that there does not exist a *functorial decomposition* of a finitely generated abelian group into a free part and a finite part. Namely, there is no natural transformation $\sigma : F \rightarrow \mathrm{Id}$, such that for every M , the homomorphism $\sigma_M : F(M) \rightarrow M$ splits the sequence (0.1.3). (Hint: find a counterexample.)

We know that for a free finitely generated abelian group F there is reflexivity, i.e. $\eta_F : F \rightarrow D(D(F))$ is an isomorphism. But for a finite abelian group T we have

$$D(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z}) = 0.$$

Thus, for a $M \in \mathbf{Ab}_f$ with nonzero torsion subgroup T , reflexivity fails: $\eta_M : M \rightarrow D(D(M))$ is not an isomorphism.

On the other hand, for an abelian group M we can define another sort of dual:

$$D'(M) := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

There is a natural transformation $\eta' : \text{Id} \rightarrow D' \circ D'$. For a finite abelian group T the homomorphism $\eta'_T : T \rightarrow D'(D'(T))$ is an isomorphism; this can be seen by decomposing T into cyclic groups, and for a finite cyclic group it is clear. So D' is a duality for finite abelian groups. (We may view the abelian group \mathbb{Q}/\mathbb{Z} as the group of roots of 1 in \mathbb{C} , via the exponential function; and then D' becomes *Pontryagin Duality*.)

But for a finitely generated free abelian group F we get $D'(D'(F)) = \widehat{F}$, the profinite completion of F . So once more this is not a good duality for all finitely generated abelian groups.

We could try to be more clever and “patch” the two dualities D and D' , into something that we will call $D \oplus D'$. This looks pleasing at first – but then we recall that the decomposition $M \cong T \oplus F$ of a finitely generated group is not functorial, so that $D \oplus D'$ can't be a functor.

This is where the *derived category* enters. For any commutative ring A there is the derived category $\mathbf{D}(\text{Mod } A)$. Here is a very quick explanation of it.

Recall that a *complex* of A -modules is a diagram

$$(0.1.4) \quad M = (\cdots \rightarrow M^{-1} \xrightarrow{d_M^{-1}} M^0 \xrightarrow{d_M^0} M^1 \rightarrow \cdots)$$

in the category $\text{Mod } A$. Namely the M^i are A -modules, and the d_M^i are homomorphisms. The condition is that $d_M^{i+1} \circ d_M^i = 0$. We sometimes write $M = \{M^i\}_{i \in \mathbb{Z}}$. The collection $d_M = \{d_M^i\}_{i \in \mathbb{Z}}$ is called the *differential* (or the coboundary operator) of M .

Given a second complex

$$N = (\cdots \rightarrow N^{-1} \xrightarrow{d_N^{-1}} N^0 \xrightarrow{d_N^0} N^1 \rightarrow \cdots),$$

a *homomorphism of complexes* $\phi : M \rightarrow N$ is a collection $\phi = \{\phi^i\}_{i \in \mathbb{Z}}$ of homomorphisms $\phi^i : M^i \rightarrow N^i$ in $\text{Mod } A$ satisfying

$$\phi^{i+1} \circ d_M^i = d_N^i \circ \phi^i.$$

The resulting category is denoted by $\mathbf{C}(\text{Mod } A)$.

The i -th *cohomology* of the complex M is

$$H^i(M) := \frac{\text{Ker}(d_M^i)}{\text{Im}(d_M^{i-1})} \in \text{Mod } A.$$

A homomorphism $\phi : M \rightarrow N$ in $\mathbf{C}(\text{Mod } A)$ induces homomorphisms

$$H^i(\phi) : H^i(M) \rightarrow H^i(N)$$

in $\text{Mod } A$. We call ϕ a *quasi-isomorphism* if all the homomorphisms $H^i(\phi)$ are isomorphisms.

The derived category $\mathbf{D}(\text{Mod } A)$ is the localization of $\mathbf{C}(\text{Mod } A)$ with respect to the quasi-isomorphisms. This means that $\mathbf{D}(\text{Mod } A)$ has the same objects as $\mathbf{C}(\text{Mod } A)$. There is a functor

$$Q : \mathbf{C}(\text{Mod } A) \rightarrow \mathbf{D}(\text{Mod } A)$$

that is the identity of objects, and it sends quasi-isomorphisms to isomorphisms. Furthermore, any morphism in $\mathbf{D}(\mathbf{Mod} A)$ can be written as a fraction:

$$Q(\phi) \circ Q(\psi)^{-1},$$

where ϕ is a morphism in $\mathbf{C}(\mathbf{Mod} A)$, and ψ is a quasi-morphism in $\mathbf{C}(\mathbf{Mod} A)$.

A single A -module M^0 can be viewed as a complex M concentrated in degree 0:

$$(0.1.5) \quad M = (\cdots \rightarrow 0 \xrightarrow{0} M^0 \xrightarrow{0} 0 \rightarrow \cdots).$$

This turns out to be a fully faithful embedding

$$(0.1.6) \quad \mathbf{Mod} A \rightarrow \mathbf{D}(\mathbf{Mod} A).$$

The essential image of this embedding is the full subcategory of $\mathbf{D}(\mathbf{Mod} A)$ on the complexes M whose cohomology is concentrated in degree 0 (i.e. $H^i(M) = 0$ for all $i \neq 0$). In this way we have *enlarged* the category of A -modules.

Here is a very important kind of quasi-isomorphism. Suppose M is a module and

$$(0.1.7) \quad \cdots \rightarrow P^{-2} \xrightarrow{d_P^{-2}} P^{-1} \xrightarrow{d_P^{-1}} P^0 \xrightarrow{\epsilon} M \rightarrow 0$$

is a free resolution of it. We can view M as a complex concentrated in degree 0, by the embedding (0.1.6). Let P be the complex

$$P = (\cdots \rightarrow P^{-2} \xrightarrow{d_P^{-2}} P^{-1} \xrightarrow{d_P^{-1}} P^0 \rightarrow 0 \rightarrow \cdots),$$

concentrated in nonpositive degrees. Then ϵ becomes a morphism of complexes

$$\epsilon : P \rightarrow M$$

with trivial components in nonzero degrees, and the exactness of the sequence (0.1.7) says that ϵ is actually a quasi-isomorphism. Thus

$$Q(\epsilon) : P \rightarrow M$$

is an isomorphism in $\mathbf{D}(\mathbf{Mod} A)$.

Let us now return to $A = \mathbb{Z}$. The functor $D = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ from $\mathbf{Mod} \mathbb{Z}$ to itself has a *right derived functor*

$$RD = \text{RHom}_{\mathbb{Z}}(-, \mathbb{Z}),$$

which is a contravariant *triangulated functor*

$$RD : \mathbf{D}(\mathbf{Mod} \mathbb{Z}) \rightarrow \mathbf{D}(\mathbf{Mod} \mathbb{Z}).$$

And there is a natural transformation of triangulated functors

$$\eta : \text{Id} \rightarrow RD \circ RD.$$

Here is the way to calculate the value of the functor RD on a finitely generated abelian group M . Let us choose a free resolution of M like in (0.1.7). To be easy on ourselves, we can take it to be of this form:

$$P = (\cdots \rightarrow 0 \rightarrow P^{-1} \xrightarrow{d_P^{-1}} P^0 \rightarrow 0 \rightarrow \cdots) = (\cdots \rightarrow 0 \rightarrow \mathbb{Z}^{r_1} \xrightarrow{d} \mathbb{Z}^{r_0} \rightarrow 0 \rightarrow \cdots),$$

where $r_0, r_1 \in \mathbb{N}$ and d is a matrix of integers. Because $Q(\epsilon) : P \rightarrow M$ is an isomorphism in $\mathbf{D}(\mathbf{Mod} \mathbb{Z})$, it suffices to calculate $RD(P)$.

It is known that $RD(P) = D(P)$ for bounded complexes of free modules, where $D(P)$ is calculated term by term. Thus

$$RD(P) = D(P) = \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) = (\cdots \rightarrow 0 \rightarrow \mathbb{Z}^{r_0} \xrightarrow{d^*} \mathbb{Z}^{r_1} \rightarrow 0 \cdots),$$

a complex concentrated in degrees 0 and 1, with the transpose matrix d^* as its differential.

Because $RD(P) = D(P)$ is itself a bounded complex of free modules, its derived dual is

$$RD(RD(P)) = D(D(P)) = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z}), \mathbb{Z}).$$

The canonical morphism

$$\eta_P : P \rightarrow D(D(P))$$

in $\mathbf{C}(\operatorname{Mod} \mathbb{Z})$ is an isomorphism in this case, because P^0 and P^{-1} are finite rank free modules. Therefore

$$\eta_M : M \rightarrow RD(RD(M))$$

is an isomorphism in $\mathbf{D}(\operatorname{Mod} \mathbb{Z})$.

We see that *RD is a duality that holds for all finitely generated \mathbb{Z} -modules!*

Here is the connection between the derived duality RD and the “classical” dualities D and D' . Take a finitely generated abelian group M , with short exact sequence (0.1.1). There are functorial isomorphisms

$$H^0(RD(M)) \cong \operatorname{Ext}_{\mathbb{Z}}^0(M, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong D(M)$$

and

$$H^1(RD(M)) \cong \operatorname{Ext}_{\mathbb{Z}}^1(M, \mathbb{Z}) \cong D'(M).$$

The cohomologies $H^i(RD(M))$ vanish for $i \neq 0, 1$.

Note that $D(M) \cong D(F)$ and $D'(M) \cong D'(T)$. We see that if M is neither free nor finite, then $H^0(RD(M))$ and $H^1(RD(M))$ are both nonzero; so that the complex $D(M)$ is not isomorphic to an object of $\operatorname{Mod} \mathbb{Z}$, under the embedding (0.1.6).

This sort of duality holds for *many noetherian commutative rings* A . But the formula for the duality functor

$$RD : \mathbf{D}(\operatorname{Mod} A) \rightarrow \mathbf{D}(\operatorname{Mod} A)$$

is somewhat different – it is

$$RD(M) := \operatorname{RHom}_A(M, R),$$

where $R \in \mathbf{D}(\operatorname{Mod} A)$ is a *dualizing complex*. Such a dualizing complex is unique (up to a degree translation and tensoring with an invertible module).

Interestingly, the structure of the dualizing complex R depends on the geometry of the ring A (i.e. of the affine scheme $\operatorname{Spec} A$). If A is a regular ring (like \mathbb{Z}) then $R = A$ is dualizing. If A is Cohen-Macaulay (and $\operatorname{Spec} A$ is connected) then R is a single A -module. But if A is a more complicated ring, then R must live in several degrees.

Example 0.1.8. Consider the affine algebraic variety $X \subseteq \mathbf{A}_{\mathbb{R}}^3$ which is the union of a plane and a line, with coordinate ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 \cdot t_1, t_3 \cdot t_2).$$

See figure 1. The dualizing complex R must live in two adjacent degrees; namely there is some i s.t. $H^i(R)$ and $H^{i+1}(R)$ are nonzero.

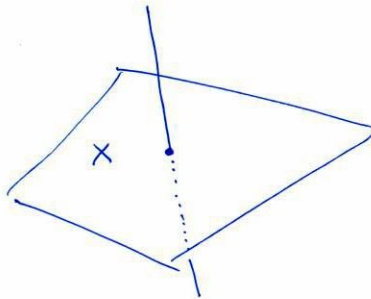


FIGURE 1. An algebraic variety that is connected but not equidimensional, and hence not Cohen-Macaulay.

One can also talk about dualizing complexes over *noncommutative rings*. (This is a favorite topic of mine!)

0.2. Synopsis of the Book. Here is a section-by-section description of the material in the book (the first half only).

Sections 1-2. These sections are pretty much a review of the standard material on categories and functors (especially abelian categories and additive functors) that is needed for the book. A reader who is familiar with this material can skip these sections. We do recommend looking at our notational convention, that are spelled out in Convention 1.2.1.

Section 3. A good understanding of *DG algebra* (“DG” is short for “differential graded”) is essential in our approach to derived categories. We aim to study both the derived category $\mathbf{D}(\mathbf{M})$ of an abelian category \mathbf{M} , and the derived category $\mathbf{D}(A)$ of DG modules over a DG ring A . In order to accomplish this, we introduce a new concept, that combines both these setups: the category $\mathbf{C}(A, \mathbf{M})$ of *DG A -modules in \mathbf{M}* . See Subsection 3.7.

Actually, our methods can be expanded to handle the DG category $\mathbf{C}(A, \mathbf{M})$ of DG A -modules in \mathbf{M} , where A is a DG category (rather than a DG ring as above). This includes as a special case ($\mathbf{M} = \mathbf{Ab}$) the category $\mathbf{C}(A)$ of DG A -modules, in the sense of Keller; see Remark 3.7.7. We have decided to stick to the less general setup $\mathbf{C}(A, \mathbf{M})$ for these reasons:

- (1) The treatment is much more streamlined and intuitive.
- (2) Virtually all DG categories that occur in practice (in algebra and algebraic geometry) are full subcategories of $\mathbf{C}(A, \mathbf{M})$, for suitable A and \mathbf{M} . A noteworthy instance is derived Morita theory for schemes (see Section 15.4), that fits nicely within our framework.

There do not exist (to our knowledge) detailed textbook references for DG algebra (by which we mean DG rings, DG modules, DG categories, DG functors and related constructions). Therefore we have included a lot of basic material in this section. Moreover, we present a new treatment of translations and cones, using the “little t operator”, following our paper [Ye9]. Among other things, we prove

(in Theorem 3.8.7) that the translation functor T of $\mathbf{C}(A, \mathbf{M})$ is a DG functor, and $t : \text{Id} \rightarrow T$ is a degree -1 morphism of DG functors from $\mathbf{C}(A, \mathbf{M})$ to itself.

Section 4. This section consists mostly of new material, some of it implicit in the paper [BoKa] on *pretriangulated DG categories*.

Inside the DG category $\mathbf{C}(A, \mathbf{M})$ there is the *strict category* $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, that has all the objects, but its morphisms are the degree 0 cocycles. Any morphism $\phi : M \rightarrow N$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ gives rise to a *standard triangle*

$$M \xrightarrow{\phi} N \xrightarrow{e_\phi} \text{Cone}(\phi) \xrightarrow{p_\phi} T(M)$$

in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$.

Consider a DG functor

$$(0.2.1) \quad F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N}),$$

where A and B are DG rings, and \mathbf{M} and \mathbf{N} are abelian categories. In Theorem 4.2.3 we show that there is a canonical isomorphism of DG functors

$$(0.2.2) \quad \tau_F : F \circ T \xrightarrow{\cong} T \circ F$$

called the *translation isomorphism*. Then, in Theorem 4.3.7, we prove that F sends standard triangles in the $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ to standard triangles in $\mathbf{C}_{\text{str}}(B, \mathbf{N})$.

We end this section with several examples of DG functors. These examples are prototypes – they can be easily extended to other setups.

Section 5. We start with the theory of *pretriangulated categories* and *triangulated functors*, following mainly [RD]. Because the *octahedral axiom* plays no role in our approach, we exclude it from the discussion, and this is the reason we do not talk about triangulated categories. In Subsection 5.4 we prove that the homotopy category $\mathbf{K}(A, \mathbf{M})$ is pretriangulated.

We conclude this section with Theorem 5.4.13. It says that given a DG functor F as in (0.2.1), with translation isomorphism τ_F from (0.2.2), the T -additive functor

$$(F, \tau_F) : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{K}(B, \mathbf{N})$$

is triangulated. This is possibly a new result (unifying well-known yet disparate examples).

Section 6. In this section we take a close look at *localization of categories*. We give a detailed proof of the theorem on Ore localization (also known as noncommutative localization). We then prove that the localization \mathbf{K}_S of a pretriangulated category \mathbf{K} at a multiplicatively closed set of cohomological origin S is a left and right Ore localization, the category \mathbf{K}_S is pretriangulated, and the localization functor $Q : \mathbf{K} \rightarrow \mathbf{K}_S$ is triangulated.

Section 7. In the case of the pretriangulated category $\mathbf{K}(A, \mathbf{M})$, and the quasi-isomorphisms $\mathbf{S}(A, \mathbf{M})$ in it, we get the *derived category*

$$\mathbf{D}(A, \mathbf{M}) := \mathbf{K}(A, \mathbf{M})_{\mathbf{S}(A, \mathbf{M})},$$

and the triangulated localization functor

$$Q : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{D}(A, \mathbf{M}).$$

We look at the full subcategories $\mathbf{K}^*(A, \mathbf{M})$ of $\mathbf{K}(A, \mathbf{M})$ corresponding to boundedness conditions \star , and prove that their localizations with respect to quasi-isomorphisms embed fully faithfully in $\mathbf{D}(A, \mathbf{M})$. We also prove that the obvious functor $\mathbf{M} \rightarrow \mathbf{D}(\mathbf{M})$ is fully faithful.

Section 8. In this section we talk about *derived functors*. To make the definitions of the derived functors precise, we introduce some *2-categorical notation* here.

The setting is general: we start from a triangulated functor $F : \mathbf{K} \rightarrow \mathbf{E}$ between pretriangulated categories, and a denominator set of cohomological origin $S \subseteq \mathbf{K}$. A *right derived functor* of F is a pair $(\mathbf{R}F, \eta)$, where $\mathbf{R}F : \mathbf{K}_S \rightarrow \mathbf{E}$ is a triangulated functor, and $\eta : F \rightarrow \mathbf{R}F \circ Q$ is a morphism of triangulated functors. The pair $(\mathbf{R}F, \eta)$ has a universal property, making it unique up to a unique isomorphism. The *left derived functor* $(\mathbf{L}F, \eta)$ is defined similarly.

We provide a general existence theorem for derived functors. For the right derived functor we assume the existence of a pretriangulated category $\mathbf{J} \subseteq \mathbf{K}$ that is “right F -acyclic”. Likewise for the left derived functor. This is the original result from [RD], but our proof is much more detailed.

Section 9. Here we specialize the general existence theorem from Section 8 to the case of the pretriangulated categories $\mathbf{K}^*(A, \mathbf{M})$, for a DG ring A , and abelian category \mathbf{M} and a boundedness condition \star . We define *K-injective DG modules*, and show they can be used to present any right derived functor (if there are enough of them). We also define *K-projective* and *K-flat DG modules*, and explain how they are used.

Section 10. In this section we prove existence of K-injective, K-projective and K-flat resolutions in several important cases of $\mathbf{C}^*(A, \mathbf{M})$:

- K-projective resolutions in $\mathbf{C}^-(\mathbf{M})$, where \mathbf{M} is an abelian category with enough projectives. This is classical (i.e. it is already in [RD]).
- K-projective resolutions in $\mathbf{C}(A)$, where A is any DG ring. This includes $\mathbf{C}(\text{Mod } A)$, the category of unbounded complexes of modules over a ring A .
- K-injective resolutions in $\mathbf{C}^+(\mathbf{M})$, where \mathbf{M} is an abelian category with enough injectives. This is classical too.
- K-injective resolutions in $\mathbf{C}(A)$, where A is any DG ring. This includes $\mathbf{C}(\text{Mod } A)$ for any ring A .

Our proofs are explicit, and we use limits of complexes cautiously (since this is known to be a pitfall).

This ends the first half of the book. As mentioned before, the second half is yet to be written.

0.3. Recommended Bibliography. For further discussion of categories (and the related set theory), functors, and classical homological algebra, see the books [Mac2], [HiSt], [Rot], [GeMa], [KaSc1], [KaSc2], [Ne], and [We].

Derived categories are treated in [RD] (the original reference), and in the last five books in the previous list. None of these references has emphasis on DG categories as the background out of which derived categories arise; indeed, most of these books do not even mention DG algebra.

Sources for algebraic geometry and modern differential geometry are [Har] and [KaSc1]. For commutative ring theory see the books [Eis], [Mat] and [AlKl]. For noncommutative ring theory see [Row] and [Rot].

Almost everything can be found in the evolving online reference [SP].

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1. BASIC FACTS ON CATEGORIES

1.1. Set Theory. In this book we will not try to be precise about issues of set theory. The blanket assumption is that we are given a *Grothendieck universe* \mathbf{U} . This is a “large” infinite set. A *small set*, or a \mathbf{U} -small set, is a set S that is an element of \mathbf{U} . We want all the products $\prod_{i \in I} S_i$ and disjoint unions $\coprod_{i \in I} S_i$, with I and S_i small sets, to be small sets too. (This requirement is not crucial for us, and it is more a matter of convenience. When dealing with higher categories, one usually needs a hierarchy of universes anyhow.) We assume that the axiom of choice holds in \mathbf{U} .

A \mathbf{U} -category is a category \mathbf{C} whose set of objects $\text{Ob}(\mathbf{C})$ is a subset of \mathbf{U} , and for every $C, D \in \text{Ob}(\mathbf{C})$ the set of morphisms $\text{Hom}_{\mathbf{C}}(C, D)$ is small. If $\text{Ob}(\mathbf{C})$ is also small, then \mathbf{C} is called a *small category*. See [SGA 4] or [KaSc2, Section 1.1]. Another approach, involving “sets” vs “classes”, can be found in [Ne].

We denote by \mathbf{Set} the category of all small sets. So $\text{Ob}(\mathbf{Set}) = \mathbf{U}$, and \mathbf{Set} is a \mathbf{U} -category. A group (or a ring, etc.) is called small if its underlying set is small. We denote by \mathbf{Grp} , \mathbf{Ab} , \mathbf{Ring} and $\mathbf{Ring}_{\mathbf{C}}$ the categories of small groups, small abelian groups, small rings and small commutative rings respectively. For a small ring A we denote by $\mathbf{Mod} A$ the category of all small left A -modules.

By default we work with \mathbf{U} -categories, and from now on \mathbf{U} will remain implicit. The one exception is when we deal with localization of categories, where we shall briefly encounter a set theoretical issue; but for most interesting cases this issue has an easy solution.

1.2. Notation. Let \mathbf{C} be a category. We often write $C \in \mathbf{C}$ as an abbreviation for $C \in \text{Ob}(\mathbf{C})$. For an object C , its identity automorphism is denoted by id_C . The identity functor of \mathbf{C} is denoted by $\text{Id}_{\mathbf{C}}$.

The opposite category of \mathbf{C} is \mathbf{C}^{op} . It has the same objects as \mathbf{C} , but the morphism sets are

$$\text{Hom}_{\mathbf{C}^{\text{op}}}(C_0, C_1) := \text{Hom}_{\mathbf{C}}(C_1, C_0),$$

and composition is reversed. The identity functor of \mathbf{C} can be viewed as a contravariant functor $\text{Id}_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{C}^{\text{op}}$. A contravariant functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is the same as a covariant functor $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$. Since we prefer dealing only with covariant functors, we make the following convention:

Convention 1.2.1. By default all functors will be covariant, unless explicitly mentioned otherwise.

We will try to keep the following font and letter conventions:

- $f : C \rightarrow D$ is a morphism between objects in a category.
- $F : \mathbf{C} \rightarrow \mathbf{D}$ is a functor between categories.
- $\eta : F \rightarrow G$ is morphism of functors (i.e. a natural transformation) between functors $F, G : \mathbf{C} \rightarrow \mathbf{D}$.
- $f, \phi, \alpha : M \rightarrow N$ are morphisms between objects in an abelian category \mathbf{M} .
- $F : \mathbf{M} \rightarrow \mathbf{N}$ is an additive functor between abelian categories.
- The derived category of an abelian category \mathbf{M} is $\mathbf{D}(\mathbf{M})$.
- If \mathbf{M} is a module category, and $M \in \text{Ob}(\mathbf{M})$, then elements of M will be denoted by m, n, m_i, \dots

1.3. Epimorphisms and Monomorphisms. Let \mathcal{C} be a category. Recall that a morphism $f : C \rightarrow D$ in \mathcal{C} is called an *isomorphism* if there is a morphism $g : D \rightarrow C$ such that $f \circ g = \text{id}_D$ and $g \circ f = \text{id}_C$. The morphism g is called the *inverse* of f , it is unique (if it exists), and it is denoted by f^{-1} . An isomorphism is often denoted by this shape of arrow: $f : C \xrightarrow{\sim} D$.

A morphism $f : C \rightarrow D$ in \mathcal{C} is called an *epimorphism* if it has the right cancellation property: for any $g, g' : D \rightarrow E$, $g \circ f = g' \circ f$ implies $g = g'$. An epimorphism is often denoted by this shape of arrow: $f : C \twoheadrightarrow D$.

A morphism $f : C \rightarrow D$ is called a *monomorphism* if it has the left cancellation property: for any $g, g' : E \rightarrow C$, $f \circ g = f \circ g'$ implies $g = g'$. A monomorphism is often denoted by this shape of arrow: $f : C \rightarrowtail D$.

Example 1.3.1. In \mathbf{Set} the monomorphisms are the injections, and the epimorphisms are the surjections. A morphism $f : C \rightarrow D$ in \mathbf{Set} that is both a monomorphism and an epimorphism is an isomorphism. The same holds in the category $\mathbf{Mod} A$ of left modules over a ring A .

This example could be misleading, because the property of being an epimorphism is often not preserved by forgetful functors, as the next exercise shows.

Exercise 1.3.2. Consider the category of rings \mathbf{Ring} . (All rings have units, and ring homomorphisms are unital.) Show that the forgetful functor $\mathbf{Ring} \rightarrow \mathbf{Set}$ respects monomorphisms, but it does not respect epimorphisms. (Hint: show that the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism in \mathbf{Ring} .)

By a *subobject* of an object $C \in \mathcal{C}$ we mean a monomorphism $f : C' \rightarrowtail C$ in \mathcal{C} . We sometimes write $C' \subseteq C$ in this situation, but this is only notational (and does not mean inclusion of sets). We say that two subobjects $f_0 : C'_0 \rightarrowtail C$ and $f_1 : C'_1 \rightarrowtail C$ of C are *isomorphic* if there is an isomorphism $g : C'_0 \xrightarrow{\sim} C'_1$ such that $f_1 \circ g = f_0$.

Likewise, by a *quotient* of C we mean an epimorphism $g : C \twoheadrightarrow C''$ in \mathcal{C} . There is an analogous notion of isomorphic quotients.

Exercise 1.3.3. Let \mathcal{C} be a category, and let C be an object of \mathcal{C} .

- (1) Suppose $f_0 : C'_0 \rightarrowtail C$ and $f_1 : C'_1 \rightarrowtail C$ are subobjects of C . Show that there is at most one morphism $g : C'_0 \rightarrow C'_1$ such that $f_1 \circ g = f_0$; and if g exists, then it is a monomorphism.
- (2) Show that isomorphism is an equivalence relation on the set of subobjects of C . Show that the set of equivalence classes of subobjects of C is partially ordered by “inclusion”. (Ignore set-theoretical issues.)
- (3) Formulate and prove the analogous statements for quotient objects.

An *initial object* in a category \mathcal{C} is an object $C_0 \in \mathcal{C}$, such that for every object $C \in \mathcal{C}$ there is exactly one morphism $C_0 \rightarrow C$. Thus the set $\text{Hom}_{\mathcal{C}}(C_0, C)$ is a singleton. A *terminal object* in \mathcal{C} is an object $C_\infty \in \mathcal{C}$, such that for every object $C \in \mathcal{C}$ there is exactly one morphism $C \rightarrow C_\infty$.

Definition 1.3.4. A *zero object* in a category \mathcal{C} is an object which is both initial and terminal.

Initial, terminal and zero objects are unique up to unique isomorphisms (but they need not exist).

Example 1.3.5. In \mathbf{Set} , \emptyset is an initial object, and any singleton is a terminal object. There is no zero object.

Example 1.3.6. In $\mathbf{Mod} A$, any trivial module (with only the zero element) is a zero object, and we denote this module by 0 . This is allowed, since any other zero module is uniquely isomorphic to it.

1.4. Products and Coproducts. Let \mathcal{C} be a category. By a *collection of objects of \mathcal{C}* indexed by a (small) set I , we mean a function $I \rightarrow \mathbf{Ob}(\mathcal{C})$, $i \mapsto C_i$. We usually denote this collection like this: $\{C_i\}_{i \in I}$.

Given a collection $\{C_i\}_{i \in I}$ of objects of \mathcal{C} , its *product* is a pair $(C, \{p_i\}_{i \in I})$ consisting of an object $C \in \mathcal{C}$, and a collection $\{p_i\}_{i \in I}$ of morphisms $p_i : C \rightarrow C_i$, called *projections*. The pair $(C, \{p_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : D \rightarrow C_i$, there is a unique morphism $f : D \rightarrow C$ s.t. $f_i = p_i \circ f$. Of course if a product $(C, \{p_i\}_{i \in I})$ exists, then it is unique up to a unique isomorphism; and we usually write $\prod_{i \in I} C_i := C$, leaving the projection morphisms implicit.

Example 1.4.1. In \mathbf{Set} and $\mathbf{Mod} A$ all products exist, and they are the usual cartesian products.

For a collection $\{C_i\}_{i \in I}$ of objects of \mathcal{C} , their *coproduct* is a pair $(C, \{e_i\}_{i \in I})$, consisting of an object C and a collection $\{e_i\}_{i \in I}$ of morphisms $e_i : C_i \rightarrow C$, called *embeddings*. The pair $(C, \{e_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : C_i \rightarrow D$, there is a unique morphism $f : C \rightarrow D$ s.t. $f_i = f \circ e_i$. If a coproduct $(C, \{e_i\}_{i \in I})$ exists, then it is unique up to a unique isomorphism; and we write $\coprod_{i \in I} C_i := C$, leaving the embeddings implicit.

Example 1.4.2. In \mathbf{Set} the coproduct is the disjoint union. In $\mathbf{Mod} A$ the coproduct is the direct sum.

1.5. Equivalence of Categories. Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence* if there exists a functor $G : \mathcal{D} \rightarrow \mathcal{C}$, and isomorphisms of functors (i.e. natural isomorphisms) $G \circ F \xrightarrow{\sim} \text{Id}_{\mathcal{C}}$ and $F \circ G \xrightarrow{\sim} \text{Id}_{\mathcal{D}}$. Such a functor G is called a *quasi-inverse* of F , and it is unique up to isomorphism (if it exists), and it is denoted by F^{-1} .

The functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *full* (resp. *faithful*) if every $C_0, C_1 \in \mathcal{C}$ the function

$$F : \text{Hom}_{\mathcal{C}}(C_0, C_1) \rightarrow \text{Hom}_{\mathcal{D}}(F(C_0), F(C_1))$$

is surjective (resp. injective).

We know that $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence iff these two conditions hold:

- (i) F is essentially surjective on objects. This means that for every $D \in \mathcal{D}$ there is some $C \in \mathcal{C}$ and an isomorphism $F(C) \xrightarrow{\sim} D$.
- (ii) F is fully faithful (i.e. full and faithful).

Exercise 1.5.1. If you are not sure about the last claim (characterization of equivalences), then prove it. (Hint: use the axiom of choice to construct a quasi-inverse of F .)

Example 1.5.2. Let \mathcal{C} and \mathcal{D} be categories. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called an *isomorphism of categories* if it is bijective on sets of objects and on sets of morphisms. It is clear that an isomorphism of categories is an equivalence. If F is an isomorphism of categories, then it has an inverse isomorphism $F^{-1} : \mathcal{D} \rightarrow \mathcal{C}$, which is unique. In practice, it is quite rare to find an isomorphism of categories.

1.6. Bifunctors. Let \mathbf{C} and \mathbf{D} be categories. Their product is the category $\mathbf{C} \times \mathbf{D}$ defined as follows: the set of objects is

$$\text{Ob}(\mathbf{C} \times \mathbf{D}) := \text{Ob}(\mathbf{C}) \times \text{Ob}(\mathbf{D}).$$

The sets of morphisms are

$$\text{Hom}_{\mathbf{C} \times \mathbf{D}}((C_0, D_0), (C_1, D_1)) := \text{Hom}_{\mathbf{C}}(C_0, C_1) \times \text{Hom}_{\mathbf{D}}(D_0, D_1).$$

The composition is

$$(f_1, g_1) \circ (f_0, g_0) := (f_1 \circ f_0, g_1 \circ g_0),$$

and the identity morphisms are $(\text{id}_C, \text{id}_D)$.

A *bifunctor*

$$F : \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{E}$$

is by definition a functor from the product category $\mathbf{C} \times \mathbf{D}$ to \mathbf{E} . We say “bifunctor” because it is a functor of two arguments: $F(C, D) \in \mathbf{E}$. This will be especially useful when considering additive categories, because then we can talk about “bi-additive bifunctors”.

1.7. Representable Functors. Let \mathbf{C} be a category and $C \in \mathbf{C}$ an object. We get a functor

$$Y_C : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}, \quad Y_C := \text{Hom}_{\mathbf{C}}(-, C),$$

called the *Yoneda functor*. This functor sends an object C' to the set $\text{Hom}_{\mathbf{C}}(C', C)$, and a morphism $\psi : C' \rightarrow C''$ in \mathbf{C} to the function

$$Y_C(\psi) := \text{Hom}(\psi, \text{id}_C) : \text{Hom}_{\mathbf{C}}(C'', C) \rightarrow \text{Hom}_{\mathbf{C}}(C', C).$$

Now suppose we are given a morphism $\phi : C_0 \rightarrow C_1$ in \mathbf{C} . There is a morphism of functors (a natural transformation)

$$Y_\phi := \text{Hom}_{\mathbf{C}}(-, \phi) : Y_{C_0} \rightarrow Y_{C_1}.$$

Here is the first formulation of the *Yoneda Lemma*.

Proposition 1.7.1 (Yoneda Lemma v1). *Let \mathbf{C} be a category, let $C_0, C_1 \in \mathbf{C}$ be objects, and let $\eta : Y_{C_0} \rightarrow Y_{C_1}$ be a morphism of functors $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$.*

- (1) *There exists a unique morphism $\phi : C_0 \rightarrow C_1$ in \mathbf{C} such that $Y_\phi = \eta$.*
- (2) *If $\eta : Y_{C_0} \rightarrow Y_{C_1}$ is an isomorphism of functors, then $\phi : C_0 \rightarrow C_1$ is an isomorphism in \mathbf{C} .*

See [KaSc2, Section 1.4] for a proof. The proof is not hard, but it is very confusing.

A functor $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ is called *representable* if there is an isomorphism of functors $f : F \xrightarrow{\cong} Y_C$ for some object $C \in \mathbf{C}$. By Proposition 1.7.1 the pair (C, f) is unique up to a unique isomorphism (if it exists). Note that the isomorphism of sets $f_C : F(C) \xrightarrow{\cong} Y_C(C)$ gives a special element $\tilde{f} \in F(C)$ such that $f_C(\tilde{f}) = \text{id}_C$.

Here is a fancier way to state this result. Consider the category $\text{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Set})$, whose objects are the functors $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$, and whose morphisms are the morphisms of functors. There is a set-theoretic difficulty here: the sets of objects and morphisms of $\text{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Set})$ are too big (unless \mathbf{C} is a small category); so this is not a U-category, and we must enlarge the universe.

Proposition 1.7.2 (Yoneda Lemma v2). *The Yoneda functor*

$$Y : \mathbf{C} \rightarrow \mathbf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}), \quad C \mapsto Y_C, \quad \phi \mapsto Y_\phi$$

is fully faithful.

In other words, the Yoneda Lemma says that the functor Y is an equivalence from \mathbf{C} to the category of representable functors $\mathbf{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$.

Dually, any $C \in \mathbf{C}$ gives rise to a functor

$$Y'_C : \mathbf{C} \rightarrow \mathbf{Set}, \quad Y'_C := \mathrm{Hom}_{\mathbf{C}}(C, -).$$

The identity automorphism id_C is a special element of the set $Y'_C(C)$.

A functor $F : \mathbf{C} \rightarrow \mathbf{Set}$ is called *corepresentable* if $F \cong Y'_C$ for some object $C \in \mathbf{C}$. The object C is said to corepresent the functor F . The dual Yoneda Lemma (v2) says that the functor Y' is an equivalence from \mathbf{C}^{op} to the category of corepresentable functors $\mathbf{C} \rightarrow \mathbf{Set}$.

2. ABELIAN CATEGORIES AND ADDITIVE FUNCTORS

The concept of *abelian category* is an extremely useful abstraction of module categories, introduced by Grothendieck in 1957. Before defining it (in Definition 2.3.8), we need some preparation.

2.1. Linear Categories.

Definition 2.1.1. Let \mathbb{K} be a commutative ring. A \mathbb{K} -linear category is a category \mathbf{M} , endowed with a \mathbb{K} -module structure on each of the sets of morphisms $\text{Hom}_{\mathbf{M}}(M_0, M_1)$. The condition is this:

- For all $M_0, M_1, M_2 \in \mathbf{M}$ the composition function

$$\text{Hom}_{\mathbf{M}}(M_1, M_2) \times \text{Hom}_{\mathbf{M}}(M_0, M_1) \rightarrow \text{Hom}_{\mathbf{M}}(M_0, M_2)$$

$$(\phi_2, \phi_1) \mapsto \phi_2 \circ \phi_1$$

is \mathbb{K} -bilinear.

If $\mathbb{K} = \mathbb{Z}$, we say that \mathbf{M} is a *linear category*.

Let \mathbb{K} be a commutative ring. By *central \mathbb{K} -ring* we mean a ring A , with a ring homomorphism $\mathbb{K} \rightarrow A$, such that the image of \mathbb{K} is inside the center of A . (Many texts would call such A a “unital associative \mathbb{K} -algebra”.)

Example 2.1.2. Let \mathbb{K} be any nonzero commutative ring, and let n be a positive integer. Then the ring of matrices $A := \text{Mat}_n(\mathbb{K})$ is a central \mathbb{K} -ring.

Proposition 2.1.3. Let \mathbf{M} be a \mathbb{K} -linear category.

- (1) For any object $M \in \mathbf{M}$, the set

$$\text{End}_{\mathbf{M}}(M) := \text{Hom}_{\mathbf{M}}(M, M),$$

with its given addition operation, and with the operation of composition, is a central \mathbb{K} -ring.

- (2) For any two objects $M_0, M_1 \in \mathbf{M}$, the set $\text{Hom}_{\mathbf{M}}(M_0, M_1)$, with its given addition operation, and with the operations of composition, is a left module over the ring $\text{End}_{\mathbf{M}}(M_1)$, and a right module over the ring $\text{End}_{\mathbf{M}}(M_0)$. Furthermore, these left and right actions commute with each other.

Proof. Exercise. □

This result can be reversed:

Example 2.1.4. Let A be a central \mathbb{K} -ring. Define a category \mathbf{M} like this: there is a single object M , and its set of morphisms is $\text{Hom}_{\mathbf{M}}(M, M) := A$. Composition in \mathbf{M} is the multiplication of A . Then \mathbf{M} is a \mathbb{K} -linear category.

Because of the above, in a linear category \mathbf{M} , we often denote the identity automorphism of an object M by $1_M := \text{id}_M \in \text{End}_{\mathbf{M}}(M)$.

For a central \mathbb{K} -ring A , the opposite ring A^{op} has the same \mathbb{K} -module structure as A , but the multiplication is reversed.

Exercise 2.1.5. Let A be a nonzero ring. Let $P, Q \in \text{Mod } A$ be distinct free A -modules of rank 1.

- (1) Prove that there is a ring isomorphism $\text{End}_{\text{Mod } A}(P) \cong A^{\text{op}}$. Is this ring isomorphism canonical?
- (2) Let \mathbf{M} be the full subcategory of $\text{Mod } A$ on the set of objects $\{P, Q\}$. Compare the linear category \mathbf{M} to the ring of matrices $\text{Mat}_2(A^{\text{op}})$.

2.2. Additive Categories.

Definition 2.2.1. An *additive category* is a linear category \mathbf{M} satisfying these conditions:

- (i) \mathbf{M} has a zero object 0 .
- (ii) \mathbf{M} has finite coproducts.

Observe that $\text{Hom}_{\mathbf{M}}(M, N) \neq \emptyset$ for any $M, N \in \mathbf{M}$, since this is an abelian group. Also

$$\text{Hom}_{\mathbf{M}}(M, 0) = \text{Hom}_{\mathbf{M}}(0, M) = 0,$$

the zero abelian group. We denote the unique arrows $0 \rightarrow M$ and $M \rightarrow 0$ also by 0 . So the numeral 0 has a lot of meanings; but they are (hopefully) clear from the contexts. The coproduct in a linear category \mathbf{M} is usually denoted by \oplus ; cf. Example 1.4.2.

Example 2.2.2. Let A be a \mathbb{K} -central ring. The category $\text{Mod } A$ is a \mathbb{K} -linear additive category. The full subcategory $\mathbf{F} \subseteq \text{Mod } A$ on the free modules is also additive.

Proposition 2.2.3. Let \mathbf{M} be a linear category. Let $\{M_i\}_{i \in I}$ be a finite collection of objects of \mathbf{M} , and assume the coproduct $M = \bigoplus_{i \in I} M_i$ exists, with embeddings $e_i : M_i \rightarrow M$.

- (1) For any i let $p_i : M \rightarrow M_i$ be the unique morphism s.t. $p_i \circ e_i = 1_{M_i}$, and $p_i \circ e_j = 0$ for $j \neq i$. Then $(M, \{p_i\}_{i \in I})$ is a product of the collection $\{M_i\}_{i \in I}$.
- (2) $\sum_{i \in I} e_i \circ p_i = 1_M$.

Proof. Exercise. □

Part (1) directly implies:

Corollary 2.2.4. An additive category has finite products.

Definition 2.2.5. Let \mathbf{M} be an additive category, and let \mathbf{N} be a full subcategory of \mathbf{M} . We say that \mathbf{N} is a *full additive subcategory* of \mathbf{M} if \mathbf{N} contains the zero object, and is closed under finite direct sums.

Exercise 2.2.6. In the situation of Definition 2.2.5, show that the category \mathbf{N} is itself additive.

Example 2.2.7. Consider the linear category \mathbf{M} from Example 2.1.4, built from a ring A . It does not have a zero object (unless the ring A is the zero ring), so it is not additive.

A more puzzling question is this: Does \mathbf{M} have finite direct sums? This turns out to be equivalent to whether or not $A \cong A \oplus A$ as right A -modules. To see why, choose a fully faithful additive functor $F : \mathbf{M} \rightarrow \text{Mod } A^{\text{op}}$, that sends the unique object $M \in \mathbf{M}$ to a rank 1 free right A -module P . (We identify right A -modules with left A^{op} -modules.) Compare to Exercise 2.1.5.

Let $I := \{1, 2\}$, and let $\{M_i\}_{i \in I}$ be the only possible collection in \mathbf{M} indexed by I (i.e. $M_i = M$). If there is a coproduct in \mathbf{M} , then it must be $M_1 \oplus M_2 \cong M$. According to Proposition 2.4.2, we get

$$P \oplus P \cong F(M_1) \oplus F(M_2) \cong F(M) \cong P$$

in $\text{Mod } A^{\text{op}}$.

One can show that when A is nonzero and commutative, or nonzero and noetherian, then $A \not\cong A \oplus A$ in $\text{Mod } A^{\text{op}}$. On the other hand, if we take a field \mathbb{K} , and a countable rank \mathbb{K} -module N , then $A := \text{End}_{\mathbb{K}}(N)$ will satisfy $A \cong A \oplus A$.

Proposition 2.2.8. *Let \mathcal{M} be a linear category, and $N \in \mathcal{M}$. The following conditions are equivalent:*

- (i) *The ring $\text{End}_{\mathcal{M}}(N)$ is trivial.*
- (ii) *N is a zero object of \mathcal{M} .*

Proof. (ii) \Rightarrow (i): Since the set $\text{End}_{\mathcal{M}}(N)$ is a singleton, it must be the trivial ring ($1 = 0$).

(i) \Rightarrow (ii): If the ring $\text{End}_{\mathcal{M}}(N)$ is trivial, then all left and right modules over it must be trivial. Now use Proposition 2.1.3(2). \square

2.3. Abelian Categories.

Definition 2.3.1. Let \mathcal{M} be an additive category, and let $f : M \rightarrow N$ be a morphism in \mathcal{M} . A *kernel* of f is a pair (K, k) , consisting of an object $K \in \mathcal{M}$ and a morphism $k : K \rightarrow M$, with these properties:

- (i) $f \circ k = 0$.
- (ii) If $k' : K' \rightarrow M$ is a morphism in \mathcal{M} such that $f \circ k' = 0$, then there is a unique morphism $g : K' \rightarrow K$ such that $k' = k \circ g$.

In other words, the object K represents the functor $\mathcal{M}^{\text{op}} \rightarrow \text{Ab}$,

$$K' \mapsto \{k' \in \text{Hom}_{\mathcal{M}}(K', M) \mid f \circ k' = 0\}.$$

The kernel of f is of course unique up to a unique isomorphism (if it exists), and we denote it by $\text{Ker}(f)$. Sometimes $\text{Ker}(f)$ refers only to the object K , and other times it refers only to the morphism k ; as usual, this should be clear from the context.

Definition 2.3.2. Let \mathcal{M} be an additive category, and let $f : M \rightarrow N$ be a morphism in \mathcal{M} . A *cokernel* of f is a pair (C, c) , consisting of an object $C \in \mathcal{M}$ and a morphism $c : N \rightarrow C$, with these properties:

- (i) $c \circ f = 0$.
- (ii) If $c' : N \rightarrow C'$ is a morphism in \mathcal{M} such that $c' \circ f = 0$, then there is a unique morphism $g : C \rightarrow C'$ such that $c' = g \circ c$.

In other words, the object C corepresents the functor $\mathcal{M} \rightarrow \text{Ab}$,

$$C' \mapsto \{c' \in \text{Hom}_{\mathcal{M}}(N, C') \mid c' \circ f = 0\}.$$

The cokernel of f is of course unique up to a unique isomorphism (if it exists), and we denote it by $\text{Coker}(f)$. Sometimes $\text{Coker}(f)$ refers only to the object C , and other times it refers only to the morphism c ; as usual, this should be clear from the context.

Example 2.3.3. In $\text{Mod } A$ all kernels and cokernels exist. Given $f : M \rightarrow N$, the kernel is $k : K \rightarrow M$, where

$$K := \{m \in M \mid f(m) = 0\},$$

and the k is the inclusion. The cokernel is $c : N \rightarrow C$, where $C := N/f(M)$, and c is the canonical projection.

Proposition 2.3.4. *Let \mathbf{M} be an additive category, and let $f : M \rightarrow N$ be a morphism in \mathbf{M} .*

- (1) *If $k : K \rightarrow M$ is a kernel of f , then k is a monomorphism.*
- (2) *If $c : N \rightarrow C$ is a cokernel of f , then c is an epimorphism.*

Proof. Exercise. □

Definition 2.3.5. Assume the additive category \mathbf{M} has kernels and cokernels. Let $f : M \rightarrow N$ be a morphism in \mathbf{M} .

- (1) Define the *image* of f to be

$$\text{Im}(f) := \text{Ker}(\text{Coker}(f)).$$

- (2) Define the *coimage* of f to be

$$\text{Coim}(f) := \text{Coker}(\text{Ker}(f)).$$

The image is familiar, but the coimage is not. The next diagram should help. We start with a morphism $f : M \rightarrow N$ in \mathbf{M} . The kernel and cokernel of f fit into this diagram:

$$K \xrightarrow{k} M \xrightarrow{f} N \xrightarrow{c} C.$$

Inserting $\alpha := \text{Coker}(k) = \text{Coim}(f)$ and $\beta := \text{Ker}(c) = \text{Im}(f)$ we get the following commutative diagram (solid arrows):

$$(2.3.6) \quad \begin{array}{ccccccc} K & \xrightarrow{k} & M & \xrightarrow{f} & N & \xrightarrow{c} & C \\ & \searrow \alpha & \downarrow & \searrow \gamma & \uparrow \beta & \nearrow 0 & \\ & & M' & \xrightarrow{f'} & N' & & \end{array}$$

Since $c \circ f = 0$ there is a unique morphism γ making the diagram commutative. Now $\beta \circ \gamma \circ k = f \circ k = 0$; and β is a monomorphism; so $\gamma \circ k = 0$. Hence there is a unique morphism $f' : M' \rightarrow N'$ making the diagram commutative. We conclude that $f : M \rightarrow N$ induces a morphism

$$(2.3.7) \quad f' : \text{Coim}(f) \rightarrow \text{Im}(f).$$

Definition 2.3.8. An *abelian category* is an additive category \mathbf{M} with these extra properties:

- (i) All morphisms in \mathbf{M} admit kernels and cokernels.
- (ii) For any morphism $f : M \rightarrow N$ in \mathbf{M} , the induced morphism f' in equation (2.3.7) is an isomorphism.

Here is a less precise but (maybe) easier to remember way to state property (ii). Because $M' = \text{Coker}(\text{Ker}(f))$ and $N' = \text{Ker}(\text{Coker}(f))$, we see that

$$(2.3.9) \quad \text{Coker}(\text{Ker}(f)) = \text{Ker}(\text{Coker}(f)).$$

From now on we forget all about the coimage.

Exercise 2.3.10. For any ring A , prove that the category $\text{Mod } A$ is abelian.

This includes the category $\mathbf{Ab} = \text{Mod } \mathbb{Z}$, from which the name derives.

Definition 2.3.11. Let \mathbf{M} be an abelian category, and let \mathbf{N} be a full subcategory of \mathbf{M} . We say that \mathbf{N} is a *full abelian subcategory* of \mathbf{M} if \mathbf{N} is closed under finite direct sums, kernels and cokernels.

Exercise 2.3.12. In the situation of Definition 2.3.11, the category \mathbf{N} is itself abelian.

Example 2.3.13. Let \mathbf{M}_1 be the category of finitely generated abelian groups, and let \mathbf{M}_0 be the category of finite abelian groups. Then \mathbf{M}_1 is a full abelian subcategory of \mathbf{Ab} , and \mathbf{M}_0 is a full abelian subcategory of \mathbf{M}_1 .

Exercise 2.3.14. Let \mathbf{N} be the full subcategory of \mathbf{Ab} whose objects are the finitely generated free abelian groups. It is an additive subcategory of \mathbf{Ab} (since it is closed under direct sums).

- (1) Show that \mathbf{N} is closed under kernels in \mathbf{Ab} .
- (2) Show that \mathbf{N} is not closed under cokernels in \mathbf{Ab} , so it is not a full abelian subcategory of \mathbf{Ab} .
- (3) Show that \mathbf{N} has cokernels (not the same as those of \mathbf{Ab}). Still, it fails to be an abelian category.

Exercise 2.3.15. The category \mathbf{Grp} is not linear of course. Still, it does have a zero object (the trivial group). Show that \mathbf{Grp} has kernels and cokernels, but condition (ii) of Definition 2.3.8 fails.

Exercise 2.3.16. Let \mathbf{Hilb} be the category of Hilbert spaces over \mathbb{C} . The morphisms are the continuous \mathbb{C} -linear homomorphisms. Show that \mathbf{Hilb} is a \mathbb{C} -linear additive category with kernels and cokernels, but it is not an abelian category.

Exercise 2.3.17. Let A be a ring. Show that A is *left noetherian* iff the category $\mathbf{Mod}_f A$ of finitely generated left modules is a full abelian subcategory of $\mathbf{Mod} A$.

Example 2.3.18. Let (X, \mathcal{A}) be a ringed space; namely X is a topological space and \mathcal{A} is a sheaf of rings on X (see [Har, Sections II.1-2]). We denote by $\mathbf{PMod} \mathcal{A}$ the category of presheaves of left \mathcal{A} -modules on X . This is an abelian category. Given a morphism $f : \mathcal{M} \rightarrow \mathcal{N}$ in $\mathbf{PMod} \mathcal{A}$, its kernel is the presheaf \mathcal{K} defined by

$$\Gamma(U, \mathcal{K}) := \text{Ker}(f : \Gamma(U, \mathcal{M}) \rightarrow \Gamma(U, \mathcal{N}))$$

on every open set $U \subseteq X$. The cokernel is the presheaf \mathcal{C} defined by

$$\Gamma(U, \mathcal{C}) := \text{Coker}(f : \Gamma(U, \mathcal{M}) \rightarrow \Gamma(U, \mathcal{N})).$$

Now let $\mathbf{Mod} \mathcal{A}$ be the full subcategory of $\mathbf{PMod} \mathcal{A}$ consisting of sheaves. It is a full additive subcategory of $\mathbf{PMod} \mathcal{A}$, closed under kernels. We know that $\mathbf{Mod} \mathcal{A}$ is not closed under cokernels inside $\mathbf{PMod} \mathcal{A}$, and hence it is not a full abelian subcategory.

However $\mathbf{Mod} \mathcal{A}$ is itself an abelian category, but with different cokernels. Indeed, for a morphism $f : \mathcal{M} \rightarrow \mathcal{N}$ in $\mathbf{Mod} \mathcal{A}$, its cokernel $\text{Coker}_{\mathbf{Mod} \mathcal{A}}(f)$ is the sheafification of the presheaf $\text{Coker}_{\mathbf{PMod} \mathcal{A}}(f)$.

Here is a general result about abelian categories.

Theorem 2.3.19 (Freyd & Mitchell). *Let \mathbf{M} be a small abelian category. Then \mathbf{M} is equivalent to a full abelian subcategory of $\mathbf{Mod} A$, for a suitable ring A .*

This means that most of the time we can pretend that $\mathbf{M} \subseteq \mathbf{Mod} A$. This is a helpful heuristic; although in practice it is not a very useful fact.

Proposition 2.3.20. (1) *Let \mathbf{M} be an additive category. Then the opposite category \mathbf{M}^{op} is also additive.*

- (2) Let \mathbf{M} be an abelian category. Then the opposite category \mathbf{M}^{op} is also abelian.

Proof. (1) First note that

$$\text{Hom}_{\mathbf{M}^{\text{op}}}(M, N) = \text{Hom}_{\mathbf{M}}(N, M),$$

so this is an abelian group. The bilinearity of the composition in \mathbf{M}^{op} is clear, and the zero objects are the same. Existence of finite coproducts in \mathbf{M}^{op} is because of existence of finite products in \mathbf{M} ; see Proposition 2.2.3(1).

(2) \mathbf{M}^{op} has kernels and cokernels, since $\text{Ker}_{\mathbf{M}^{\text{op}}}(f) = \text{Coker}_{\mathbf{M}}(f)$ and vice versa. Also the symmetric condition (ii) of Definition 2.3.8 holds. \square

Proposition 2.3.21. Let $f : M \rightarrow N$ be a morphism in an abelian category \mathbf{M} .

- (1) f is a monomorphism iff $\text{Ker}(f) = 0$.
- (2) f is an epimorphism iff $\text{Coker}(f) = 0$.
- (3) f is an isomorphism iff it is both a monomorphism and an epimorphism.

Proof. Exercise. \square

2.4. Additive Functors.

Definition 2.4.1. Let \mathbf{M} and \mathbf{N} be \mathbb{K} -linear categories. A functor $F : \mathbf{M} \rightarrow \mathbf{N}$ is called a \mathbb{K} -linear functor if for every $M_0, M_1 \in \mathbf{M}$ the function

$$F : \text{Hom}_{\mathbf{M}}(M_0, M_1) \rightarrow \text{Hom}_{\mathbf{N}}(F(M_0), F(M_1))$$

is a \mathbb{K} -linear homomorphism.

A \mathbb{Z} -linear functor is also called an *additive functor*.

Additive functors commute with finite direct sums. More precisely:

Proposition 2.4.2. Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be an additive functor between linear categories, let $\{M_i\}_{i \in I}$ be a finite collection of objects of \mathbf{M} , and assume that the direct sum $(M, \{e_i\}_{i \in I})$ of the collection $\{M_i\}_{i \in I}$ exists in \mathbf{M} . Then $(F(M), \{F(e_i)\}_{i \in I})$ is a direct sum of the collection $\{F(M_i)\}_{i \in I}$ in \mathbf{N} .

Exercise 2.4.3. Prove Proposition 2.4.2. (Hint: use Proposition 2.2.3.)

Note that the proposition above also talks about finite products, because of Proposition 2.2.3.

Example 2.4.4. Let $A \rightarrow B$ be a ring homomorphism. The corresponding forgetful functor

$$F : \text{Mod } B \rightarrow \text{Mod } A$$

(also called restriction of scalars) is additive. The functor

$$G : \text{Mod } A \rightarrow \text{Mod } B$$

defined by $G(M) := B \otimes_A M$, called extension of scalars, is also additive.

Proposition 2.4.5. Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be an additive functor between linear categories. Then:

- (1) For any $M \in \mathbf{M}$ the function

$$F : \text{End}_{\mathbf{M}}(M) \rightarrow \text{End}_{\mathbf{N}}(F(M))$$

is a ring homomorphism.

- (2) For any $M_0, M_1 \in \mathbf{M}$ the function

$$F : \text{Hom}_{\mathbf{M}}(M_0, M_1) \rightarrow \text{Hom}_{\mathbf{N}}(F(M_0), F(M_1))$$

is a homomorphism of left $\text{End}_{\mathbf{M}}(M_1)$ -modules, and of right $\text{End}_{\mathbf{M}}(M_0)$ -modules.

- (3) If M is a zero object of \mathbf{M} , then $F(M)$ is a zero object of \mathbf{N} .

Proof. (1) By Definition 2.4.1 the function F respects addition. By the definition of a functor, it respects multiplication and units.

(2) Immediate from the definitions, like (1).

(3) Combine part (1) with Proposition 2.2.8. \square

Definition 2.4.6. Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be an additive functor between abelian categories.

- (1) F is called *left exact* if it commutes with kernels. Namely for any morphism $\phi : M_0 \rightarrow M_1$ in \mathbf{M} , with kernel $k : K \rightarrow M_0$, the morphism $F(k) : F(K) \rightarrow F(M_0)$ is a kernel of $F(\phi) : F(M_0) \rightarrow F(M_1)$.
- (2) F is called *right exact* if it commutes with cokernels. Namely for any morphism $\phi : M_0 \rightarrow M_1$ in \mathbf{M} , with cokernel $c : M_1 \rightarrow C$, the morphism $F(c) : F(M_1) \rightarrow F(C)$ is a cokernel of $F(\phi) : F(M_0) \rightarrow F(M_1)$.
- (3) F is called *exact* if it is both left exact and right exact.

This is illustrated in the following diagrams. Suppose $\phi : M_0 \rightarrow M_1$ is a morphism in \mathbf{M} , with kernel K and cokernel C . Applying F to the diagram

$$K \xrightarrow{k} M_0 \xrightarrow{\phi} M_1 \xrightarrow{c} C$$

we get the solid arrows in

$$\begin{array}{ccccccc} F(K) & \xrightarrow{F(k)} & F(M_0) & \xrightarrow{F(\phi)} & F(M_1) & \xrightarrow{F(c)} & F(C) \\ & \searrow \psi & \uparrow & & \downarrow & & \nearrow \chi \\ & & \text{Ker}_{\mathbf{N}}(F(\phi)) & & \text{Coker}_{\mathbf{N}}(F(\phi)) & & \end{array}$$

Because \mathbf{N} is abelian, we get the vertical dashed arrows: the kernel and cokernel of $F(\phi)$. The slanted dashed arrows exist and are unique because $F(\phi) \circ F(k) = 0$ and $F(c) \circ F(\phi) = 0$. Left exactness requires ψ to be an isomorphism, and right exactness requires χ to be an isomorphism.

Definition 2.4.7. Let \mathbf{M} be an abelian category. An *exact sequence* in \mathbf{M} is a diagram

$$\cdots \rightarrow M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \rightarrow \cdots$$

(finite or infinite on either side), such that for every index i for which ϕ_{i-1} and ϕ_i are both defined, the composition $\phi_i \circ \phi_{i-1}$ is zero, and the induced morphism $\text{Im}(\phi_{i-1}) \rightarrow \text{Ker}(\phi_i)$ is an isomorphism.

As usual, a *short exact sequence* is one of the form

$$(2.4.8) \quad 0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0.$$

Proposition 2.4.9. Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be an additive functor between abelian categories.

- (1) The functor F is left exact iff for every short exact sequence (2.4.8) in \mathbf{M} , the sequence

$$0 \rightarrow F(M_0) \rightarrow F(M_1) \rightarrow F(M_2)$$

is exact in \mathbf{N} .

- (2) The functor F is right exact iff for every short exact sequence (2.4.8) in \mathbf{M} , the sequence

$$F(M_0) \rightarrow F(M_1) \rightarrow F(M_2) \rightarrow 0$$

is exact in \mathbf{N} .

Exercise 2.4.10. Prove Proposition 2.4.9. (Hint: $M_0 \cong \text{Ker}(M_1 \rightarrow M_2)$ etc.)

Example 2.4.11. Let A be a commutative ring, and let M be a fixed A -module. Define functors $F, G : \text{Mod } A \rightarrow \text{Mod } A$ and $H : (\text{Mod } A)^{\text{op}} \rightarrow \text{Mod } A$ like this: $F(N) := M \otimes_A N$, $G(N) := \text{Hom}_A(M, N)$ and $H(N) := \text{Hom}_A(N, M)$. Then F is right exact, and G and H are left exact.

Proposition 2.4.12. Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be an additive functor between abelian categories. If F is an equivalence then it is exact.

Proof. We will prove that F respects kernels; the proof for cokernels is similar. Take a morphism $\phi : M_0 \rightarrow M_1$ in \mathbf{M} , with kernel K . We have this diagram (solid arrows):

$$\begin{array}{ccccc} M & & & & \\ \downarrow \psi & \searrow \theta & & & \\ K & \xrightarrow{k} & M_0 & \xrightarrow{\phi} & M_1 \end{array}$$

Applying F we obtain this diagram (solid arrows):

$$\begin{array}{ccccc} N = F(M) & & & & \\ \downarrow F(\psi) & \searrow \bar{\theta} & & & \\ F(K) & \xrightarrow{F(k)} & F(M_0) & \xrightarrow{F(\phi)} & F(M_1) \end{array}$$

in \mathbf{N} . Suppose $\bar{\theta} : N \rightarrow F(M_0)$ is a morphism in \mathbf{N} s.t. $F(\phi) \circ \bar{\theta} = 0$. Since F is essentially surjective on objects, there is some $M \in \mathbf{M}$ with an isomorphism $\alpha : F(M) \xrightarrow{\sim} N$. After replacing N with $F(M)$ and $\bar{\theta}$ with $\bar{\theta} \circ \alpha$, we can assume that $N = F(M)$.

Now since F is fully faithful, there is a unique $\theta : M \rightarrow M_0$ s.t. $F(\theta) = \bar{\theta}$; and $\phi \circ \theta = 0$. So there is a unique $\psi : M \rightarrow K$ s.t. $\theta = k \circ \psi$. It follows that $F(\psi) : F(M) \rightarrow F(K)$ is the unique morphism s.t. $\bar{\theta} = F(k) \circ F(\psi)$. \square

Here is a result that could afford another proof of the previous proposition.

Proposition 2.4.13. Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be an additive functor between linear categories. Assume F is an equivalence, with quasi-inverse G . Then $G : \mathbf{N} \rightarrow \mathbf{M}$ is an additive functor.

Exercise 2.4.14. Prove Proposition 2.4.13.

2.5. Projective Objects. In this subsection \mathbf{M} is an abelian category.

A *splitting* of an epimorphism $\psi : M \rightarrow M''$ in \mathbf{M} is a morphism $\alpha : M'' \rightarrow M$ s.t. $\psi \circ \alpha = 1_{M''}$. A splitting of a monomorphism $\phi : M' \rightarrow M$ is a morphism $\beta : M \rightarrow M'$ s.t. $\beta \circ \phi = 1_{M'}$. A splitting of a short exact sequence

$$(2.5.1) \quad 0 \rightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \rightarrow 0$$

is a splitting of the epimorphism ψ , or equivalently a splitting of the monomorphism ϕ . The short exact sequence is said to be *split* if it has some splitting.

Exercise 2.5.2. Show how to get from a splitting of ϕ to a splitting of ψ , and vice versa. Show how any of those gives rise to an isomorphism $M \cong M' \oplus M''$.

Definition 2.5.3. An object $P \in \mathbf{M}$ is called a *projective object* if for any morphism $\gamma : P \rightarrow N$ and any *epimorphism* $\psi : M \rightarrow N$, there exists a morphism $\tilde{\gamma} : P \rightarrow M$ such that $\psi \circ \tilde{\gamma} = \gamma$.

This is described in the following commutative diagram in \mathbf{M} :

$$\begin{array}{ccc} & & P \\ & \swarrow \tilde{\gamma} & \downarrow \gamma \\ M & \xrightarrow{\psi} & N \end{array}$$

Proposition 2.5.4. The following conditions are equivalent for $P \in \mathbf{M}$:

- (i) P is projective.
- (ii) The additive functor

$$\mathrm{Hom}_{\mathbf{M}}(P, -) : \mathbf{M} \rightarrow \mathbf{Ab}$$

is exact.

- (iii) Any short exact sequence (2.5.1) with $M'' = P$ is split.

Proof. Exercise. □

Definition 2.5.5. We say \mathbf{M} has enough projectives if every $M \in \mathbf{M}$ admits an epimorphism $P \rightarrow M$ from a projective object P .

Exercise 2.5.6. Let A be a ring.

- (1) Prove that an A -module P is projective iff it is a direct summand of a free module; i.e. $P \oplus P' \cong Q$ for some module P' and free module Q .
- (2) Prove that the category $\mathbf{Mod} A$ has enough projectives.

Exercise 2.5.7. Let \mathbf{M} be the category of finite abelian groups. Prove that the only projective object in \mathbf{M} is 0. So \mathbf{M} does not have enough projectives. (Hint: use Proposition 2.5.4.)

Example 2.5.8. Consider the scheme $X := \mathbf{P}_{\mathbb{K}}^1$, the projective line over a field \mathbb{K} . (If the reader prefers, he/she can assume \mathbb{K} is algebraically closed, so X is a classical algebraic variety.) The structure sheaf (sheaf of functions) is \mathcal{O}_X . The category $\mathbf{Coh} \mathcal{O}_X$ of coherent \mathcal{O}_X -modules is abelian (it is a full abelian subcategory of $\mathbf{Mod} \mathcal{O}_X$, cf. Example 2.3.18). One can show that the only projective object of $\mathbf{Coh} \mathcal{O}_X$ is 0, but this is quite involved.

Let us only indicate why \mathcal{O}_X is not projective. Denote by t_0, t_1 the homogeneous coordinates of X . These belong to $\Gamma(X, \mathcal{O}_X(1))$, so each determines a homomorphism of sheaves $t_j : \mathcal{O}_X(i) \rightarrow \mathcal{O}_X(i+1)$. We get a sequence

$$0 \rightarrow \mathcal{O}_X(-2) \xrightarrow{[t_0 \ t_1]} \mathcal{O}_X(-1)^2 \xrightarrow{\begin{bmatrix} -t_1 \\ t_0 \end{bmatrix}} \mathcal{O}_X \rightarrow 0$$

in $\text{Coh } \mathcal{O}_X$, which is known to be exact. Because $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$, and $\Gamma(X, \mathcal{O}_X(-1)) = 0$, this sequence is not split.

2.6. Injective Objects. In this subsection \mathbf{M} is an abelian category.

Definition 2.6.1. An object $I \in \mathbf{M}$ is called an *injective object* if for any morphism $\gamma : M \rightarrow I$ and any *monomorphism* $\psi : M \hookrightarrow N$, there exists a morphism $\tilde{\gamma} : N \rightarrow I$ such that $\tilde{\gamma} \circ \psi = \gamma$.

This is depicted in the following commutative diagram in \mathbf{M} :

$$\begin{array}{ccc} & I & \\ \gamma \uparrow & \nearrow \tilde{\gamma} & \\ M & \xrightarrow{\psi} & N \end{array}$$

Proposition 2.6.2. *The following conditions are equivalent for $I \in \mathbf{M}$:*

- (i) *I is injective.*
- (ii) *The additive functor*

$$\text{Hom}_{\mathbf{M}}(-, I) : \mathbf{M}^{\text{op}} \rightarrow \mathbf{Ab}$$

is exact.

- (iii) *Any short exact sequence (2.5.1) with $M' = I$ is split.*

Proof. Exercise. □

Example 2.6.3. Let A be a ring. Unlike projectives, the structure of injective objects in $\text{Mod } A$ is very complicated, and not much is known (except that they exist). However if A is a commutative noetherian ring then we know this: every injective module I is a direct sum of indecomposable injective modules. And these indecomposables are parametrized by $\text{Spec } A$, the set of prime ideals of A . These facts are due to Matlis; see [RD, pages 120-122] for details.

Definition 2.6.4. We say \mathbf{M} *has enough injectives* if every $M \in \mathbf{M}$ admits a monomorphism $M \rightarrow I$ to an injective object I .

Here are a few results about injective objects. Recall that modules over a ring are always left modules by default.

Proposition 2.6.5. *Let $f : A \rightarrow B$ be a ring homomorphism, and let I be an injective A -module. Then $J := \text{Hom}_A(B, I)$ is an injective B -module.*

Proof. Note that B is a left A -module via f , and a right B -module. This makes J into a left B -module. In a formula: for $\phi \in J$ and $b, b' \in B$ we have $(b \cdot \phi)(b') = \phi(b' \cdot b)$.

Now given any $N \in \text{Mod } B$ there is an isomorphism

$$(2.6.6) \quad \text{Hom}_B(N, J) = \text{Hom}_B(N, \text{Hom}_A(B, I)) \cong \text{Hom}_A(N, I).$$

This is a natural isomorphism (of functors in N). So the functor $\text{Hom}_B(-, J)$ is exact, and hence J is injective. \square

Theorem 2.6.7 (Baer Criterion). *Let A be a ring and I an A -module. Assume that every A -module homomorphism $\mathfrak{a} \rightarrow I$ from a left ideal $\mathfrak{a} \subseteq A$ extends to a homomorphism $A \rightarrow I$. Then the module I is injective.*

Proof. Consider an A -module M , a submodule $N \subseteq M$, and a homomorphism $\gamma : N \rightarrow I$. We have to prove that γ extends to a homomorphism $M \rightarrow I$. Look at the pairs (N', γ') consisting of a submodule $N' \subseteq M$ that contains N , and a homomorphism $\gamma' : N' \rightarrow I$ that extends γ . The set of all such pairs is ordered by inclusion, and it satisfies the conditions of Zorn's Lemma. Therefore there exists a maximal pair (N', γ') . We claim that $N' = M$.

Otherwise, there is an element $m \in M$ that does not belong to N' . Define $N'' := N' + A \cdot m$, so $N' \subsetneq N'' \subseteq M$. Let

$$\mathfrak{a} := \{a \in A \mid a \cdot m \in N'\},$$

which is a left ideal of A . There is a short exact sequence

$$0 \rightarrow \mathfrak{a} \xrightarrow{\alpha} N' \oplus A \rightarrow N'' \rightarrow 0$$

of A -modules, where $\alpha(a) := (a \cdot m, -a)$. Let $\phi : \mathfrak{a} \rightarrow I$ be the homomorphism $\phi(a) := \gamma'(a \cdot m)$. By assumption, it extends to a homomorphism $\tilde{\phi} : A \rightarrow I$. We get a homomorphism

$$\gamma' + \tilde{\phi} : N' \oplus A \rightarrow I$$

that vanishes on the image of α . Thus there is an induced homomorphism $\gamma'' : N'' \rightarrow I$. This contradicts the maximality of (N', γ') . \square

Lemma 2.6.8. *The \mathbb{Z} -module \mathbb{Q}/\mathbb{Z} is injective.*

Proof. By the Baer criterion, it is enough to consider a homomorphism $\gamma : \mathfrak{a} \rightarrow \mathbb{Q}/\mathbb{Z}$ for an ideal $\mathfrak{a} = n \cdot \mathbb{Z} \subseteq \mathbb{Z}$. We may assume that $n \neq 0$. Say $\gamma(n) = r + \mathbb{Z}$ with $r \in \mathbb{Q}$. Then we can extend γ to $\tilde{\gamma} : \mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ with $\tilde{\gamma}(1) := r/n + \mathbb{Z}$. \square

Lemma 2.6.9. *Let $\{I_x\}_{x \in X}$ be a collection of injective objects of \mathbf{M} . If the product $\prod_{x \in X} I_x$ exists in \mathbf{M} , then it is an injective object.*

Proof. Exercise. \square

Theorem 2.6.10. *Let A be any ring. The category $\text{Mod } A$ has enough injectives.*

Proof. Step 1. Here $A = \mathbb{Z}$. Take any nonzero \mathbb{Z} -module M and any nonzero $m \in M$. Consider the cyclic submodule $M' := \mathbb{Z} \cdot m \subseteq M$. There is a homomorphism $\gamma' : M' \rightarrow \mathbb{Q}/\mathbb{Z}$ s.t. $\gamma'(m) \neq 0$. Indeed, if $M' \cong \mathbb{Z}$, then we take any $r \in \mathbb{Q} - \mathbb{Z}$; and if $M' \cong \mathbb{Z}/(n)$ for some $n > 0$, then we take $r := 1/n$. In either case, we define $\gamma'(m) := r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Since \mathbb{Q}/\mathbb{Z} is an injective \mathbb{Z} -module, γ' extends to a homomorphism $\gamma : M \rightarrow \mathbb{Q}/\mathbb{Z}$. By construction we have $\gamma(m) \neq 0$.

Step 2. Now A is any ring, M is any nonzero A -module, and $m \in M$ a nonzero element. Define the A -module $I := \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, which, according to Lemma 2.6.8 and Proposition 2.6.5, is an injective A -module. Let $\gamma : M \rightarrow \mathbb{Q}/\mathbb{Z}$ be a \mathbb{Z} -linear homomorphism such that $\gamma(m) \neq 0$. Such γ exists by step 1. Let $\theta : I \rightarrow \mathbb{Q}/\mathbb{Z}$ be the \mathbb{Z} -linear homomorphism that sends an element $\chi \in I$ to $\chi(1) \in \mathbb{Q}/\mathbb{Z}$. The adjunction formula (2.6.6) gives an A -module homomorphism $\psi : M \rightarrow I$ s.t. $\theta \circ \psi = \gamma$. We note that $(\theta \circ \psi)(m) = \gamma(m) \neq 0$, and hence $\psi(m) \neq 0$.

Step 3. Here A and M are arbitrary. Let I be as in step 2. For any nonzero $m \in M$ there is an A -linear homomorphism $\psi_m : M \rightarrow I$ such that $\psi_m(m) \neq 0$. For $m = 0$ let $\psi_0 : M \rightarrow I$ be an arbitrary homomorphism (e.g. $\psi_0 = 0$). Define the A -module $J := \prod_{m \in M} I$. There is a homomorphism $\psi := \prod_{m \in M} \psi_m : M \rightarrow J$, and it is easy to check that ψ is a monomorphism. By Lemma 2.6.9, J is an injective A -module. \square

Exercise 2.6.11. At the price of getting a bigger injective module, we can make the construction of injective resolutions functorial. Let $I := \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ as above. Given an A -module M , consider the set

$$X(M) := \text{Hom}_A(M, I) \cong \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

Let $J(M) := \prod_{\psi \in X(M)} I$. There is a “tautological” homomorphism $\phi_M : M \rightarrow J(M)$. Show that ϕ_M is a monomorphism, $J : M \mapsto J(M)$ is a functor, and $\phi : \text{Id} \rightarrow J$ is a natural transformation.

Is the functor $J : \text{Mod } A \rightarrow \text{Mod } A$ additive?

Example 2.6.12. Let \mathbf{N} be the category of torsion abelian groups, and \mathbf{M} the category of finite abelian groups. Then $\mathbf{N} \subseteq \mathbf{Ab}$ and $\mathbf{M} \subseteq \mathbf{N}$ are full abelian subcategories. \mathbf{M} has no projectives nor injectives except 0 (see Exercise 2.5.7 regarding projectives). The only projective in \mathbf{N} is 0. However, it can be shown that \mathbf{N} has enough injectives; see [Har, Lemma III.3.2] or [Ye1, Proposition 4.6].

Proposition 2.6.13. *If A is a left noetherian ring, then any direct sum of injective A -modules is an injective module.*

Exercise 2.6.14. Prove Proposition 2.6.13. (Hint: use the Baer criterion.)

Remark 2.6.15. Actually, the converse of Proposition 2.6.13 is also true: if every direct sum of injective A -modules is injective, then A is left noetherian. But experience tells us that this fact is not very important...

Exercise 2.6.16. Here we study injectives in the category $\mathbf{Ab} = \text{Mod } \mathbb{Z}$. By Lemma 2.6.8, the module $I := \mathbb{Q}/\mathbb{Z}$ is injective. For a (positive) prime number p , we denote by $\widehat{\mathbb{Z}}_p$ the ring of p -adic integers, and by $\widehat{\mathbb{Q}}_p$ its field of fractions (namely the p -adic completions of \mathbb{Z} and \mathbb{Q} respectively). Define the abelian group $I_p := \widehat{\mathbb{Q}}_p / \widehat{\mathbb{Z}}_p$.

- (1) Show that I_p is an injective object of \mathbf{Ab} .
- (2) Show that I_p is indecomposable (i.e. it is not the direct sum of two nonzero objects).
- (3) Show that $I \cong \bigoplus_p I_p$.
- (4) The theory tells us that there is another indecomposable injective object in \mathbf{Ab} , besides the I_p . Try to identify it.

Remark 2.6.17. Let \mathbb{K} be a field and $A := \mathbb{K}[t]$, the polynomial ring in one variable. As we very well know, the categories $\text{Mod } A$ and $\text{Mod } \mathbb{Z}$ share many properties. Let $A^* := \text{Hom}_{\mathbb{K}}(A, \mathbb{K})$, which is of course an injective A -module (because \mathbb{K} is an injective \mathbb{K} -module). The structure of A^* , as a direct sum of indecomposable injectives, was used to cook up a counterexample in [Ye5, Section 6].

The abelian category $\text{Mod } \mathcal{A}$ associated to a ringed space (X, \mathcal{A}) was introduced in Example 2.3.18.

Proposition 2.6.18. *Let (X, \mathcal{A}) be a ringed space. The category $\text{Mod } \mathcal{A}$ has enough injectives.*

Proof. Let \mathcal{M} be an \mathcal{A} -module. Take a point $x \in X$. The stalk \mathcal{M}_x is a module over the ring \mathcal{A}_x , and by Theorem 2.6.10 we can find an embedding $\phi_x : \mathcal{M}_x \rightarrow I_x$ into an injective \mathcal{A}_x -module. Let $g_x : \{x\} \rightarrow X$ be the inclusion, which we may view as a map of ringed spaces from $(\{x\}, \mathcal{A}_x)$ to (X, \mathcal{A}) . Define $\mathcal{I}_x := g_{x*}(I_x)$, which is an \mathcal{A} -module (in fact it is a constant sheaf supported on the closed set $\overline{\{x\}} \subseteq X$). The adjunction formula gives rise to a sheaf homomorphism $\psi_x : \mathcal{M} \rightarrow \mathcal{I}_x$. Since the functor $g_x^* : \text{Mod } \mathcal{A} \rightarrow \text{Mod } \mathcal{A}_x$ is exact, the adjunction formula shows that \mathcal{I}_x is an injective object of $\text{Mod } \mathcal{A}$.

Finally let $\mathcal{J} := \prod_{x \in X} \mathcal{I}_x$. This is an injective \mathcal{A} -module. There is a homomorphism $\psi := \prod_{x \in X} \psi_x : \mathcal{M} \rightarrow \mathcal{J}$ in $\text{Mod } \mathcal{A}$. This is a monomorphism, since for every point x , letting \mathcal{J}_x be the stalk of the sheaf \mathcal{J} at x , the composition $\mathcal{M}_x \xrightarrow{\psi_x} \mathcal{J}_x \xrightarrow{p_x} \mathcal{I}_x$ is the embedding $\phi_x : \mathcal{M}_x \rightarrow I_x$. \square

3. DIFFERENTIAL GRADED ALGEBRA

In this section we fix a nonzero commutative base ring \mathbb{K} . (It seems more relaxing to have a base ring \mathbb{K} around, rather than working with the universal base $\mathbb{K} = \mathbb{Z}$.) Throughout, “DG” stands for “differential graded”.

3.1. Graded Algebra. Before entering the DG world, it is good to understand the graded world.

A *graded \mathbb{K} -module* is a \mathbb{K} -module M equipped with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M^i$ into \mathbb{K} -submodules. The \mathbb{K} -module M^i is called the degree i component of M . The elements of M^i are called homogeneous elements of degree i .

Suppose M and N are graded \mathbb{K} -modules. For any integer i let

$$(M \otimes_{\mathbb{K}} N)^i := \bigoplus_{j \in \mathbb{Z}} (M^j \otimes_{\mathbb{K}} N^{i-j}).$$

Then

$$(3.1.1) \quad M \otimes_{\mathbb{K}} N = \bigoplus_{i \in \mathbb{Z}} (M \otimes_{\mathbb{K}} N)^i,$$

is a graded \mathbb{K} -module.

A \mathbb{K} -linear homomorphism $\phi : M \rightarrow N$ is said to be of degree i if $\phi(M^j) \subseteq N^{j+i}$ for all j . We denote by $\text{Hom}_{\mathbb{K}}(M, N)^i$ the \mathbb{K} -module of degree i homomorphisms $M \rightarrow N$. In other words

$$\text{Hom}_{\mathbb{K}}(M, N)^i = \prod_{j \in \mathbb{Z}} \text{Hom}_{\mathbb{K}}(M^j, N^{j+i}).$$

Then

$$(3.1.2) \quad \text{Hom}_{\mathbb{K}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{K}}(M, N)^i$$

is a graded \mathbb{K} -module. A degree 0 homomorphism $\phi : M \rightarrow N$ is sometimes called a *strict homomorphism of graded \mathbb{K} -modules*.

If M_0, M_1, M_2 are graded \mathbb{K} -modules, and $\phi_k : M_{k-1} \rightarrow M_k$ are \mathbb{K} -linear homomorphisms of degrees i_k , then $\phi_2 \circ \phi_1 : M_0 \rightarrow M_2$ is a \mathbb{K} -linear homomorphism of degree $i_1 + i_2$. The identity automorphism $1_M : M \rightarrow M$ has degree 0.

A *graded ring* is a ring A , equipped with decomposition as an abelian group $A = \bigoplus_{i \in \mathbb{Z}} A^i$, such that the unit element $1 \in A^0$, and $A^i \cdot A^j \subseteq A^{i+j}$. A *central graded \mathbb{K} -ring* is a graded ring A , together with a ring homomorphism $\mathbb{K} \rightarrow A^0$, such that the image of \mathbb{K} is central in A (i.e. $\lambda \cdot a = a \cdot \lambda$ for all $\lambda \in \mathbb{K}$ and $a \in A$). A *homomorphism of central graded \mathbb{K} -rings* $f : A \rightarrow B$ is a ring homomorphism that respects the gradings and the homomorphisms from \mathbb{K} . As always for ring homomorphisms, f must preserve units, i.e. $f(1_A) = 1_B$.

Example 3.1.3. Let M be a graded \mathbb{K} -module. Then the graded module

$$\text{End}_{\mathbb{K}}(M) := \text{Hom}_{\mathbb{K}}(M, M),$$

with the operation of composition, is a central graded \mathbb{K} -ring.

The manipulation of graded modules and homomorphisms involves the *Koszul sign rule*, that says this (in deliberately vague terms): suppose a and b are homogeneous elements of degrees i and j . Let $f(\dots, a, b, \dots)$ be a multilinear graded

expression, such that the expression $f(\dots, b, a, \dots)$ is also defined (i.e. it makes sense to transpose a and b), and for the ungraded expression f^{\natural} there is equality

$$f^{\natural}(\dots, b, a, \dots) = f^{\natural}(\dots, a, b, \dots).$$

Then

$$f(\dots, b, a, \dots) = (-1)^{ij} \cdot f(\dots, a, b, \dots).$$

The next examples ought to clarify how the Koszul sign rule is used.

Example 3.1.4. Suppose that for $k = 0, 1$ we are given graded \mathbb{K} -module homomorphisms $\phi_k : M_k \rightarrow N_k$ of degrees i_k . Then the homomorphism

$$\phi_0 \otimes \phi_1 \in \text{Hom}_{\mathbb{K}}(M_0 \otimes_{\mathbb{K}} M_1, N_0 \otimes_{\mathbb{K}} N_1)^{i_0+i_1}$$

acts on a tensor $m_0 \otimes m_1 \in M_0 \otimes_{\mathbb{K}} M_1$, with $m_k \in M_k^{j_k}$, like this:

$$(\phi_0 \otimes \phi_1)(m_0 \otimes m_1) := (-1)^{i_1 \cdot j_0} \cdot \phi_0(m_0) \otimes \phi_1(m_1) \in N_0 \otimes_{\mathbb{K}} N_1.$$

The sign is because ϕ_1 and m_0 were transposed.

Example 3.1.5. Suppose we are given graded \mathbb{K} -module homomorphisms $\phi_0 : N_0 \rightarrow M_0$ and $\phi_1 : M_1 \rightarrow N_1$ of degrees i_0 and i_1 . Then the homomorphism

$$\text{Hom}(\phi_0, \phi_1) \in \text{Hom}_{\mathbb{K}}(\text{Hom}_{\mathbb{K}}(M_0, M_1), \text{Hom}_{\mathbb{K}}(N_0, N_1))^{i_0+i_1}$$

acts on $\gamma \in \text{Hom}_{\mathbb{K}}(M_0, M_1)^j$ as follows: for an element $n_0 \in N_0^k$ we have

$$\text{Hom}(\phi_0, \phi_1)(\gamma)(n_0) := (-1)^{i_0 \cdot (i_1+j)} (\phi_1 \circ \gamma \circ \phi_0)(n_0) \in N_1^{k+i_0+i_1+j}.$$

The sign is because ϕ_0 jumped across ϕ_1 and γ .

Example 3.1.6. Let A and B be central graded \mathbb{K} -rings. Then $A \otimes_{\mathbb{K}} B$ is a central graded \mathbb{K} -ring, with multiplication

$$(a_0 \otimes b_0) \cdot (a_1 \otimes b_1) := (-1)^{i_1 \cdot j_0} \cdot (a_0 \cdot a_1) \otimes (b_0 \cdot b_1)$$

for elements $a_k \in A^{i_k}$ and $b_k \in B^{j_k}$.

Example 3.1.7. The Koszul sign rule influences the meaning of commutativity for graded rings. A graded ring A is called *weakly commutative* if $b \cdot a = (-1)^{ij} \cdot a \cdot b$ for all $a \in A^i$ and $b \in A^j$.

Remark 3.1.8. Following [Ye9], there are two variants of commutativity for DG rings. Besides the weakly commutative graded rings from the example above, there are also *strongly commutative* graded rings; the extra condition is that $a^2 = 0$ if the degree i is odd. If \mathbb{K} has characteristic 0 (i.e. it contains \mathbb{Q}), then the two variants of commutativity coincide.

Exercise 3.1.9. Let A be a central graded \mathbb{K} -ring. Elements $a \in A^i$ and $b \in A^j$ are said to graded-commute if $b \cdot a = (-1)^{ij} \cdot a \cdot b$. A homogeneous element $a \in A$ is called graded-central if it graded-commutes with all other homogeneous elements. The *graded center* of A is the \mathbb{K} -linear span of all graded-central homogeneous elements in A . Show that the graded center of A is a graded subring of A .

Let A be a central graded \mathbb{K} -ring. A *graded left A -module* is a left A -module M , equipped with a \mathbb{K} -module decomposition $M = \bigoplus_{i \in \mathbb{Z}} M^i$, such that $A^i \cdot M^j \subseteq M^{i+j}$. We can also talk about graded right A -modules, and graded bimodules. But our default option is that modules are left modules.

If M is a graded \mathbb{K} -module, A is a central graded \mathbb{K} -ring, and $f : A \rightarrow \text{End}_{\mathbb{K}}(M)$ is a graded \mathbb{K} -ring homomorphism, then M becomes a graded A -module, with action $a \cdot m := f(a)(m)$. Any graded A -module structure on M arises this way.

Lemma 3.1.10. *Let A be a central graded \mathbb{K} -ring, let M be a right graded A -module, and let N be a left graded A -module. Then the \mathbb{K} -module $M \otimes_A N$ has a direct sum decomposition*

$$M \otimes_A N = \bigoplus_{i \in \mathbb{Z}} (M \otimes_A N)^i,$$

where $(M \otimes_A N)^i$ is the \mathbb{K} -linear span of the tensors $m \otimes n$ with $m \in M^j$ and $n \in N^{i-j}$.

Proof. There is a canonical surjection of \mathbb{K} -modules

$$M \otimes_{\mathbb{K}} N \rightarrow M \otimes_A N.$$

Its kernel is the \mathbb{K} -submodule $L \subseteq M \otimes_{\mathbb{K}} N$ generated by the elements

$$(m \cdot a) \otimes n - m \otimes (a \cdot n),$$

for $m \in M^j$, $n \in N^k$ and $a \in A^l$. So L is a graded submodule of $M \otimes_{\mathbb{K}} N$, and therefore so is the quotient. Finally, by formula (3.1.1) the i -th homogeneous component of $M \otimes_A N$ is precisely $(M \otimes_A N)^i$. \square

Definition 3.1.11. Let A be a central graded \mathbb{K} -ring, and let M, N be graded A -modules. For any $i \in \mathbb{Z}$ define $\text{Hom}_A(M, N)^i$ to be the subset of $\text{Hom}_{\mathbb{K}}(M, N)^i$ consisting of the homomorphisms $\phi : M \rightarrow N$ such that

$$\phi(a \cdot m) = (-1)^{ik} \cdot a \cdot \phi(m)$$

for all $a \in A^k$. Next let

$$\text{Hom}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_A(M, N)^i.$$

Suppose \mathcal{C} is a \mathbb{K} -linear category (Definition 2.1.1). Since the composition of morphisms is \mathbb{K} -bilinear, for any triple of objects $M_0, M_1, M_2 \in \mathcal{C}$, composition can be expressed as a \mathbb{K} -linear homomorphism

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(M_1, M_2) \otimes_{\mathbb{K}} \text{Hom}_{\mathcal{C}}(M_0, M_1) &\rightarrow \text{Hom}_{\mathcal{C}}(M_0, M_2) \\ \phi_2 \otimes \phi_1 &\mapsto \phi_2 \circ \phi_1. \end{aligned}$$

We refer to it as the composition homomorphism. It will be used in the following definition.

Definition 3.1.12. A *graded \mathbb{K} -linear category* is a \mathbb{K} -linear category \mathcal{C} , endowed with a grading on each of the \mathbb{K} -modules $\text{Hom}_{\mathcal{C}}(M_0, M_1)$. The conditions are these:

- (a) For any object M , the identity automorphism 1_M has degree 0.
- (b) For any triple of objects $M_0, M_1, M_2 \in \mathcal{C}$, the composition homomorphism

$$\text{Hom}_{\mathcal{C}}(M_1, M_2) \otimes_{\mathbb{K}} \text{Hom}_{\mathcal{C}}(M_0, M_1) \rightarrow \text{Hom}_{\mathcal{C}}(M_0, M_2)$$

is a strict homomorphism of graded \mathbb{K} -modules.

In item (b) we use the graded module structure on a tensor product from equation (3.1.1). A morphism $\phi \in \text{Hom}_{\mathcal{C}}(M_0, M_1)^i$ is called a morphism of degree i .

Definition 3.1.13. Let \mathcal{C} be a graded \mathbb{K} -linear category. We define \mathcal{C}^0 to be the subcategory of \mathcal{C} on all objects, but the morphisms are only the degree 0 morphisms.

Example 3.1.14. Let A be a central graded \mathbb{K} -ring. Define $\mathbf{G}(A)$ to be the category whose objects are the graded A -modules. For $M, N \in \mathbf{G}(A)$, the set of morphisms is the graded module

$$\mathrm{Hom}_{\mathbf{G}(A)}(M, N) := \mathrm{Hom}_A(M, N)$$

from Definition 3.1.11. Then $\mathbf{G}(A)$ is a graded \mathbb{K} -linear category. The morphisms in the subcategory $\mathbf{G}^0(A)$ are the strict homomorphisms of graded A -modules.

Definition 3.1.15. Let \mathbf{C} and \mathbf{D} be graded \mathbb{K} -linear categories. A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is called a *graded \mathbb{K} -linear functor* if it satisfies this condition:

▷ For any pair of objects $M_0, M_1 \in \mathbf{C}$, the function

$$F : \mathrm{Hom}_{\mathbf{C}}(M_0, M_1) \rightarrow \mathrm{Hom}_{\mathbf{D}}(F(M_0), F(M_1))$$

is a strict homomorphism of graded \mathbb{K} -modules.

Example 3.1.16. Let A be a central graded \mathbb{K} -ring. We can view A as a category \mathbf{A} with a single object, and it is a \mathbb{K} -linear graded category. If $f : A \rightarrow B$ is a homomorphism of central graded \mathbb{K} -rings, then passing to single-object categories we get a \mathbb{K} -linear graded functor $F : \mathbf{A} \rightarrow \mathbf{B}$.

Recall that “morphism of functors” is synonymous with “natural transformation”.

Definition 3.1.17. Let $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be \mathbb{K} -linear graded functors between \mathbb{K} -linear graded categories, and let $i \in \mathbb{Z}$. A *degree i morphism of graded functors* $\eta : F \rightarrow G$ is a collection $\eta = \{\eta_M\}_{M \in \mathbf{C}}$ of morphisms

$$\eta_M \in \mathrm{Hom}_{\mathbf{D}}(F(M), G(M))^i,$$

such that for any morphism $\phi \in \mathrm{Hom}_{\mathbf{C}}(M_0, M_1)^j$, there is equality

$$G(\phi) \circ \eta_{M_0} = (-1)^{ij} \cdot \eta_{M_1} \circ F(\phi)$$

inside

$$\mathrm{Hom}_{\mathbf{D}}(F(M_0), G(M_1))^{i+j}.$$

Definition 3.1.18. Let \mathbf{M} be a \mathbb{K} -linear category. A *graded object in \mathbf{M}* is a collection $\{M^i\}_{i \in \mathbb{Z}}$ of objects $M^i \in \mathbf{M}$.

Because we did not assume that \mathbf{M} has countable direct sums, the graded objects are “external” to \mathbf{M} ; cf. Example 3.1.21.

Suppose $M = \{M^i\}_{i \in \mathbb{Z}}$ and $N = \{N^i\}_{i \in \mathbb{Z}}$ are graded objects in \mathbf{M} . For any integer i we define the \mathbb{K} -module

$$\mathrm{Hom}_{\mathbf{M}}(M, N)^i := \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{M}}(M^j, N^{j+i}).$$

We get a graded \mathbb{K} -module

$$(3.1.19) \quad \mathrm{Hom}_{\mathbf{M}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{M}}(M, N)^i.$$

Definition 3.1.20. Let \mathbf{M} be a \mathbb{K} -linear category. The *category of graded objects in \mathbf{M}* is the \mathbb{K} -linear graded category $\mathbf{G}(\mathbf{M})$, whose objects are the graded objects in \mathbf{M} , and the morphism sets are the graded modules

$$\mathrm{Hom}_{\mathbf{G}(\mathbf{M})}(M, N) := \mathrm{Hom}_{\mathbf{M}}(M, N)$$

from equation (3.1.19). The composition operation is the obvious one.

Example 3.1.21. Suppose $\mathbf{M} = \text{Mod } A$, the category of modules over a central \mathbb{K} -ring A . For any $M = \{M^i\}_{i \in \mathbb{Z}} \in \mathbf{G}(\mathbf{M})$ let $F(M) := \bigoplus_{i \in \mathbb{Z}} M^i$. Then $F(M)$ is a graded A -module, as discussed earlier, so $F(M)$ is an object of the category $\mathbf{G}(A)$ from Example 3.1.14. It is not hard to verify that

$$F : \mathbf{G}(\mathbf{M}) \rightarrow \mathbf{G}(A)$$

is an equivalence of \mathbb{K} -linear graded categories.

In the next definition we combine graded rings and linear categories, to concoct a new hybrid.

Definition 3.1.22. Let \mathbf{M} be a \mathbb{K} -linear category, and let A be a central graded \mathbb{K} -ring. A *graded A -module in \mathbf{M}* is an object $M \in \mathbf{G}(\mathbf{M})$, together with graded \mathbb{K} -ring homomorphism $f : A \rightarrow \text{End}_{\mathbf{M}}(M)$.

What the definition says is that any element $a \in A^i$ gives rise to a degree i endomorphism $f(a)$ of the graded object $M = \{M^i\}_{i \in \mathbb{Z}}$. In turn, this means that for every j , $f(a) : M^j \rightarrow M^{j+i}$ is a morphism in \mathbf{M} . The operation f satisfies $f(1_A) = 1_M$ and $f(a_1 \cdot a_2) = f(a_1) \circ f(a_2)$

Example 3.1.23. If $A = \mathbb{K}$, then $\mathbf{G}(A, \mathbf{M}) = \mathbf{G}(\mathbf{M})$; and if $\mathbf{M} = \text{Mod } \mathbb{K}$, then $\mathbf{G}(A, \mathbf{M}) = \mathbf{G}(A)$.

The next definition is a variant of Definition 3.1.11.

Definition 3.1.24. Let \mathbf{M} be a \mathbb{K} -linear category, and let A be a central graded \mathbb{K} -ring. For $M, N \in \mathbf{G}(A, \mathbf{M})$ and $i \in \mathbb{Z}$ we define $\text{Hom}_{A, \mathbf{M}}(M, N)^i$ to be the subset of $\text{Hom}_{\mathbf{M}}(M, N)^i$ consisting of the morphisms $\phi : M \rightarrow N$ such that

$$\phi \circ f_M(a) = (-1)^{ik} \cdot f_N(a) \circ \phi$$

for all $a \in A^k$. Next let

$$\text{Hom}_{A, \mathbf{M}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A, \mathbf{M}}(M, N)^i.$$

This is a graded \mathbb{K} -module.

Definition 3.1.25. Let \mathbf{M} be a \mathbb{K} -linear category, and let A be a central graded \mathbb{K} -ring. The *category of graded A -modules in \mathbf{M}* is the \mathbb{K} -linear graded category $\mathbf{G}(A, \mathbf{M})$ whose objects are the graded A -modules in \mathbf{M} , and the morphism sets are the graded \mathbb{K} -modules

$$\text{Hom}_{\mathbf{G}(A, \mathbf{M})}(M_0, M_1) := \text{Hom}_{A, \mathbf{M}}(M_0, M_1)$$

from Definition 3.1.24.

Notice that forgetting the action of A is a faithful \mathbb{K} -linear graded functor $\mathbf{G}(A, \mathbf{M}) \rightarrow \mathbf{G}(\mathbf{M})$. As in any graded category, there is the subcategory $\mathbf{G}^0(A, \mathbf{M}) \subseteq \mathbf{G}(A, \mathbf{M})$ of strict morphisms.

Exercise 3.1.26. Show that if \mathbf{M} is an abelian \mathbb{K} -linear category, then $\mathbf{G}^0(A, \mathbf{M})$ is an abelian category.

3.2. DG \mathbb{K} -modules.

Definition 3.2.1. A *DG \mathbb{K} -module* is a graded \mathbb{K} -module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, together with a \mathbb{K} -linear operator $d_M : M \rightarrow M$ of degree 1, called the differential, satisfying $d_M \circ d_M = 0$.

When there is no danger of confusion, we may write d instead of d_M .

Definition 3.2.2. Let M and N be DG \mathbb{K} -modules. A *strict homomorphism of DG \mathbb{K} -modules* is a \mathbb{K} -linear homomorphism $\phi : M \rightarrow N$ that commutes with the differentials and respects the gradings. The resulting category is denoted by $\text{DGMod}_{\text{str}} \mathbb{K}$.

Remark 3.2.3. The name “strict morphism of DG modules”, and the corresponding notation $\text{DGMod}_{\text{str}} \mathbb{K}$, are new. We introduced them to distinguish $\text{DGMod}_{\text{str}} \mathbb{K}$ from the DG category $\text{DGMod} \mathbb{K}$ that contains it; cf. Definitions 3.4.1 and 3.4.4.

Suppose M and N are DG \mathbb{K} -modules. Their tensor product $M \otimes_{\mathbb{K}} N$ was defined, as a graded module, in equation (3.1.1). We put on it the differential

$$(3.2.4) \quad d(m \otimes n) := d_M(m) \otimes n + (-1)^i \cdot m \otimes d_N(n)$$

for $m \in M^i$ and $n \in N^j$. In this way $M \otimes_{\mathbb{K}} N$ becomes a DG \mathbb{K} -module. We sometimes write $d_{M \otimes_{\mathbb{K}} N}$ for the differential.

The Hom graded module $\text{Hom}_{\mathbb{K}}(M, N)$ was introduced in equation (3.1.2). There is a differential on it:

$$(3.2.5) \quad d(\phi) := d_N \circ \phi - (-1)^i \cdot \phi \circ d_M$$

for $\phi \in \text{Hom}_{\mathbb{K}}(M, N)^i$. When we need to emphasize where d acts, we sometimes denote it by $d_{\text{Hom}_{\mathbb{K}}(M, N)}$.

Let M be a DG \mathbb{K} -module. The module of degree i cocycles of M is

$$(3.2.6) \quad Z^i(M) := \text{Ker}(d|_{M^i}) \subseteq M^i,$$

and the module of degree i coboundaries is

$$(3.2.7) \quad B^i(M) := \text{Im}(d|_{M^{i-1}}) \subseteq M^i.$$

Since $d \circ d = 0$ we have $B^i(N) \subseteq Z^i(N)$. The i -th cohomology is

$$(3.2.8) \quad H^i(M) := Z^i(M) / B^i(M).$$

These are all \mathbb{K} -modules, and in fact they are functors

$$Z^i, B^i, H^i : \text{DGMod}_{\text{str}} \mathbb{K} \rightarrow \text{Mod } \mathbb{K}.$$

Let us record the following result, whose easy proof we leave out.

Proposition 3.2.9. *For DG \mathbb{K} -modules M and N , there is equality*

$$\text{Hom}_{\text{DGMod}_{\text{str}} \mathbb{K}}(M, N) = Z^0(\text{Hom}_{\mathbb{K}}(M, N))$$

of these submodules of $\text{Hom}_{\mathbb{K}}(M, N)$.

3.3. DG Rings and Modules.

Definition 3.3.1. A *DG ring* is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, together with an operator $d_A : A \rightarrow A$ of degree 1 called the differential, satisfying the equation $d_A \circ d_A = 0$, and the graded Leibniz rule

$$d_A(a \cdot b) = d_A(a) \cdot b + (-1)^i \cdot a \cdot d_A(b)$$

for all $a \in A^i$ and $b \in A^j$.

We sometimes write d instead of d_A .

Definition 3.3.2. Let A and B be DG rings. A *homomorphism of DG rings* $f : A \rightarrow B$ is a ring homomorphism that commutes with the differentials and respects the gradings. The resulting category is denoted by \mathbf{DGRing} .

Rings are viewed as DG rings concentrated in degree 0 (and with trivial differentials). Thus the category of rings \mathbf{Ring} is a full subcategory of \mathbf{DGRing} .

Definition 3.3.3. We say that A is a *central DG \mathbb{K} -ring* if there is a given DG ring homomorphism $\mathbb{K} \rightarrow A$, whose image is central in A .

We denote by $\mathbf{DGRing}/_{\text{ce}} \mathbb{K}$ the category of central DG \mathbb{K} -rings, in which the morphisms $f : A \rightarrow B$ are the homomorphisms in \mathbf{DGRing} that respect the given structural homomorphisms from \mathbb{K} .

Proposition 3.3.4. *Let A be a central DG \mathbb{K} -ring. Then the differential d_A is \mathbb{K} -linear. In particular, the image of \mathbb{K} is contained in $Z^0(A)$.*

Proof. Exercise. □

Of course when $\mathbb{K} = \mathbb{Z}$ we have $\mathbf{DGRing}/_{\text{ce}} \mathbb{K} = \mathbf{DGRing}$.

Here are few examples of DG rings. First a silly example.

Example 3.3.5. Let A be a central graded \mathbb{K} -ring. Then A , with the trivial differential, is a central DG \mathbb{K} -ring.

Example 3.3.6. Let X be a differentiable (i.e. of type C^∞) manifold over \mathbb{R} . The de Rham complex A of X is a central DG \mathbb{R} -ring, with the wedge product and the exterior differential. See [KaSc1, Section 2.9.7] for details. In fact A is a commutative DG ring, in the sense of Remark 3.1.8.

The next example is the algebraic analogue of the previous one.

Example 3.3.7. Let C be a commutative \mathbb{K} -ring. Then the algebraic de Rham complex $A := \Omega_{C/\mathbb{K}} = \bigoplus_{p \geq 0} \Omega_{C/\mathbb{K}}^p$ is a strongly commutative DG \mathbb{K} -ring. See [Eis, Exercise 16.15] or [Mat, Section 25].

Example 3.3.8. Let M be a DG \mathbb{K} -module. Consider the DG \mathbb{K} -module

$$\text{End}_{\mathbb{K}}(M) := \text{Hom}_{\mathbb{K}}(M, M).$$

Composition of endomorphisms is an associative multiplication on $\text{End}_{\mathbb{K}}(M)$ that respects the grading, and the graded Leibniz rule holds. We see that $\text{End}_{\mathbb{K}}(M)$ is a central DG \mathbb{K} -ring.

Example 3.3.9. Let C be a commutative ring and let $c \in C$ be an element. The *Koszul complex* of c is the DG C -module $K(C; c)$ defined as follows. In degree 0 we let $K^0(C; c) := C$. In degree -1 , $K^{-1}(C; c)$ is a free C -module of rank 1, with

basis element x . All other homogeneous components are trivial. The differential d is determined by what it does to the basis element $x \in K^{-1}(C; c)$, and we let $d(x) := c \in K^0(C; c)$.

To make $K(C; c)$ into a DG ring, we treat x as an odd variable (in the sense of strongly commutative DG rings – see Remark 3.1.8). This dictates $x^2 = 0$. It is easy to verify that $K(C; c)$ is a central DG C -ring.

Example 3.3.10. Let A and B be central DG \mathbb{K} -rings. The graded ring $A \otimes_{\mathbb{K}} B$ from Example 3.1.6, with the differential (3.2.4), is a central DG \mathbb{K} -ring.

Example 3.3.11. Let C be a commutative ring and let $\mathbf{c} = (c_1, \dots, c_n)$ be a sequence of elements in C . By combining Examples 3.3.9 and 3.3.10 we obtain the Koszul complex

$$K(C; \mathbf{c}) := K(C; c_1) \otimes_C \cdots \otimes_C K(C; c_n).$$

This is a strongly commutative DG C -ring. In the classical literature the multiplicative structure of $K(C; \mathbf{c})$ has been usually ignored; see [Eis] and [Mat].

Definition 3.3.12. Let A be a central DG \mathbb{K} -ring. The *opposite DG ring* A^{op} is the same DG \mathbb{K} -module as A , but the multiplication \cdot^{op} is reversed and twisted by signs:

$$a \cdot^{\text{op}} b := (-1)^{ij} \cdot b \cdot a$$

for $a \in A^i$ and $b \in A^j$.

Definition 3.3.13. Let A be a central DG \mathbb{K} -ring. A *left DG A -module* is a graded left A -module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with an operator $d_M : M \rightarrow M$ of degree 1 called the differential, satisfying $d_M \circ d_M = 0$ and

$$d_M(a \cdot m) = d_A(a) \cdot m + (-1)^i \cdot a \cdot d_M(m)$$

for $a \in A^i$ and $m \in M^j$.

Right DG A -modules are defined likewise, but we won't deal with them much. This is because right DG A -modules are left DG modules over the opposite DG ring A^{op} . More precisely, if M is a right DG A -module, then the formula

$$(3.3.14) \quad a \cdot m := (-1)^{ij} \cdot m \cdot a,$$

for $a \in A^i$ and $m \in M^j$, makes M into a left DG A^{op} -module.

So we make this convention for the rest of the book (analogous to Convention 1.2.1):

Convention 3.3.15. By default, DG modules are *left DG modules*. In particular, a module over a ring is by default a left module.

Proposition 3.3.16. Let A be a central DG \mathbb{K} -ring, and let M be a DG \mathbb{K} -module.

- (1) Suppose $f : A \rightarrow \text{End}_{\mathbb{K}}(M)$ is a DG \mathbb{K} -ring homomorphism. Then the formula $a \cdot m := f(a)(m)$, for $a \in A^i$ and $m \in M^j$, makes M into a DG A -module.
- (2) Conversely, any DG A -module structure on M , that's compatible with the DG \mathbb{K} -module structure, arises in this way from a DG \mathbb{K} -ring homomorphism $f : A \rightarrow \text{End}_{\mathbb{K}}(M)$.

Proof. Exercise. □

Definition 3.3.17. Let M and N be DG A -modules. A *strict homomorphism of DG A -modules* is a \mathbb{K} -linear homomorphism $\phi : M \rightarrow N$ that respects the differentials, the gradings and the action of A . The resulting category is denoted by $\text{DGMod}_{\text{str}} A$.

Exercise 3.3.18. Let A be a DG ring. Show that the cocycles $Z(A) := \bigoplus_{i \in \mathbb{Z}} Z^i(A)$ are a graded subring of A , and the coboundaries $B(A) := \bigoplus_{i \in \mathbb{Z}} B^i(A)$ are a two-sided ideal of $Z(A)$. Thus the cohomology $H(A) := \bigoplus_{i \in \mathbb{Z}} H^i(A)$ is a graded ring. (Compare to Definition 3.4.4.)

Next show that given a DG A -module M , its cohomology $H(M)$ is a graded $H(A)$ -module.

Definition 3.3.19. Let A be a central DG \mathbb{K} -ring, let M be a right DG A -module, and let N be a left DG A -module. By Lemma 3.1.10, $M \otimes_A N$ is a graded \mathbb{K} -module. We make it into a DG \mathbb{K} -module with the differential from formula (3.2.4).

Definition 3.3.20. Let A be a central DG \mathbb{K} -ring, and let M, N be left DG A -modules. The graded \mathbb{K} -module $\text{Hom}_A(M, N)$ from Definition 3.1.11 is made into a DG \mathbb{K} -module with the differential from (3.2.5).

Generalizing Proposition 3.2.9, for DG A -modules M and N there is equality

$$\text{Hom}_{\text{DGMod}_{\text{str}} A}(M, N) = Z^0(\text{Hom}_A(M, N)).$$

3.4. DG Categories. In Definition 3.1.12 we saw graded categories. Here is the DG version.

Definition 3.4.1. A \mathbb{K} -linear DG category is a \mathbb{K} -linear category \mathcal{C} , endowed with a DG \mathbb{K} -module structure on each of the morphism \mathbb{K} -modules $\text{Hom}_{\mathcal{C}}(M_0, M_1)$. The conditions are these:

- (a) For any object M , the identity automorphism 1_M is a degree 0 cocycle in $\text{Hom}_{\mathcal{C}}(M, M)$.
- (b) For any triple of objects $M_0, M_1, M_2 \in \mathcal{C}$, the composition homomorphism

$$\text{Hom}_{\mathcal{C}}(M_1, M_2) \otimes_{\mathbb{K}} \text{Hom}_{\mathcal{C}}(M_0, M_1) \rightarrow \text{Hom}_{\mathcal{C}}(M_0, M_2)$$

is a strict homomorphism of DG \mathbb{K} -modules.

Definition 3.4.2. Let \mathcal{C} be a \mathbb{K} -linear DG category.

- (1) A morphism $\phi \in \text{Hom}_{\mathcal{C}}(M, N)^i$ is called a *degree i morphism*.
- (2) A morphism $\phi \in \text{Hom}_{\mathcal{C}}(M, N)$ is called a *cocycle* if $d(\phi) = 0$.
- (3) A morphism $\phi : M \rightarrow N$ in \mathcal{C} is called a *strict morphism* if it is a degree 0 cocycle.

Lemma 3.4.3. Let \mathcal{C} be a \mathbb{K} -linear DG category, and for $i = 1, 2, 3$ let $\phi_i : M_{i-1} \rightarrow M_i$ be a morphism in \mathcal{C} of degree k_i .

- (1) The morphism $\phi_2 \circ \phi_1$ has degree $k_1 + k_2$, and

$$d(\phi_2 \circ \phi_1) = d(\phi_2) \circ \phi_1 + (-1)^{k_2} \cdot \phi_2 \circ d(\phi_1).$$

- (2) If ϕ_1 and ϕ_2 are cocycles, then so is $\phi_2 \circ \phi_1$.
- (3) If ϕ_2 is a coboundary, and ϕ_1 and ϕ_3 are cocycles, then $\phi_3 \circ \phi_2 \circ \phi_1$ is a coboundary.

Proof. (1) This is just a rephrasing of item (b) in Definition 3.4.1.

(2) This is immediate from (1).

(3) Say $\phi_2 = d(\psi_2)$ for some degree $k_2 - 1$ morphism $\psi_2 : M_1 \rightarrow M_2$. Then

$$\phi_3 \circ \phi_2 \circ \phi_1 = d((-1)^{k_3} \cdot \phi_3 \circ \psi_2 \circ \phi_1).$$

□

The previous lemma makes the next definition possible.

Definition 3.4.4. Let \mathcal{C} be a \mathbb{K} -linear DG category.

- (1) The *strict category* of \mathcal{C} is the category $\text{Str}(\mathcal{C}) = \mathcal{C}_{\text{str}}$, with the same objects as \mathcal{C} , but with strict morphisms only. Thus

$$\text{Hom}_{\text{Str}(\mathcal{C})}(M, N) = Z^0(\text{Hom}_{\mathcal{C}}(M, N)).$$

- (2) The *homotopy category* of \mathcal{C} is the category $\text{Ho}(\mathcal{C})$, with the same objects as \mathcal{C} , and with morphism sets

$$\text{Hom}_{\text{Ho}(\mathcal{C})}(M, N) := H^0(\text{Hom}_{\mathcal{C}}(M, N)).$$

- (3) We denote by

$$P : \text{Str}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{C})$$

the functor which is the identity on objects, and sends a strict morphism to its homotopy class.

The categories $\text{Str}(\mathcal{C})$ and $\text{Ho}(\mathcal{C})$ are \mathbb{K} -linear. The inclusion $\text{Str}(\mathcal{C}) \rightarrow \mathcal{C}$ and $P : \text{Str}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{C})$ are \mathbb{K} -linear functors. The first is faithful (injective on morphisms), and the second is full (surjective on morphisms).

Example 3.4.5. If \mathcal{A} is a \mathbb{K} -linear DG category, then for every object $x \in \mathcal{A}$, its set of endomorphisms $A := \text{End}_{\mathcal{A}}(x)$ is a central DG \mathbb{K} -ring. Conversely, any central DG \mathbb{K} -ring A can be viewed as a \mathbb{K} -linear DG category with a single object.

Example 3.4.6. Let A be a central DG \mathbb{K} -ring. The set of DG A -modules forms a \mathbb{K} -linear DG category $\text{DGMod } A$, in which the morphism DG modules are

$$\text{Hom}_{\text{DGMod } A}(M, N) := \text{Hom}_A(M, N)$$

from Definition 3.3.20. The strict category here is

$$\text{Str}(\text{DGMod } A) = \text{DGMod}_{\text{str}} A;$$

cf. Definition 3.3.17.

Remark 3.4.7. The fact that the concept of “DG category” includes both DG rings (Example 3.4.5) and DG modules over them (Example 3.4.6) is a source of frequent confusion. See Remarks 3.4.9 and 3.7.7.

Here is the categorical version of Definition 3.3.12.

Definition 3.4.8. Let \mathcal{C} be a \mathbb{K} -linear DG category. The *opposite DG category* \mathcal{C}^{op} has the same set of objects. The morphism DG modules are

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(M, N) := \text{Hom}_{\mathcal{C}}(N, M).$$

The composition is reversed with signs:

$$\phi \circ^{\text{op}} \psi := (-1)^{ij} \cdot \psi \circ \phi$$

for composable homogeneous morphisms ϕ and ψ in \mathcal{C} of degrees i and j .

Remark 3.4.9. There is a vast theory on DG categories. See the relatively old references [Ke], [BoKa]; for more modern accounts try to search the internet (there are numerous accounts, of many flavors). In this book we shall be exclusively concerned with the categories $\mathbf{C}(A, M)$, to be introduced in Subsection 3.7, that have a lot more structure than other DG categories. See Remark 3.7.7 regarding the DG category $\mathbf{C}(A) = \mathbf{C}(A, \text{Mod } \mathbb{K})$ of left DG modules over a \mathbb{K} -linear DG category A , in the sense of [Ke].

Here is a useful result.

Proposition 3.4.10. *Let $\phi : M \rightarrow N$ be a degree i isomorphism in the \mathbb{K} -linear DG category \mathbf{C} . Assume ϕ is a cocycle, namely $d(\phi) = 0$. Then its inverse $\phi^{-1} : N \rightarrow M$ is also a cocycle.*

Proof. According the Leibniz rule (Lemma 3.4.3(1)), and the fact that 1_M is a cocycle, we have

$$0 = d(1_M) = d(\phi^{-1} \circ \phi) = d(\phi^{-1}) \circ \phi + (-1)^{-i} \cdot \phi^{-1} \circ d(\phi) = d(\phi^{-1}) \circ \phi.$$

Because ϕ is an isomorphism, we conclude that $d(\phi^{-1}) = 0$. \square

3.5. DG Functors. Here \mathbf{C} and \mathbf{D} are \mathbb{K} -linear DG categories (see Definition 3.4.1). When we forget differentials, \mathbf{C} and \mathbf{D} become \mathbb{K} -linear graded categories. So we can talk about graded functors $\mathbf{C} \rightarrow \mathbf{D}$, as in Definition 3.1.15.

The differential of the DG \mathbb{K} -module $\text{Hom}_{\mathbf{C}}(M_0, M_1)$, for objects $M_0, M_1 \in \mathbf{C}$, will be denoted by $d_{\mathbf{C}}$. Likewise in \mathbf{D} .

Recall the meaning of a strict homomorphism of DG \mathbb{K} -modules: it has degree 0 and commutes with the differentials.

Definition 3.5.1. Let \mathbf{C} and \mathbf{D} be \mathbb{K} -linear DG categories. A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is called a \mathbb{K} -linear DG functor if it satisfies this condition:

▷ For any pair of objects $M_0, M_1 \in \mathbf{C}$, the function

$$F : \text{Hom}_{\mathbf{C}}(M_0, M_1) \rightarrow \text{Hom}_{\mathbf{D}}(F(M_0), F(M_1))$$

is a strict homomorphism of DG \mathbb{K} -modules.

In other words, F is a DG functor if it is a graded functor, and

$$(3.5.2) \quad d_{\mathbf{D}} \circ F = F \circ d_{\mathbf{C}}$$

as degree 1 homomorphisms

$$\text{Hom}_{\mathbf{C}}(M_0, M_1) \rightarrow \text{Hom}_{\mathbf{D}}(F(M_0), F(M_1)).$$

Example 3.5.3. Let $f : A \rightarrow B$ be a homomorphism of central DG \mathbb{K} -rings. Define the DG categories \mathbf{C} and \mathbf{D} as follows: $\text{Ob}(\mathbf{C}) := \{x\}$, $\text{End}_{\mathbf{C}}(x) := A$, $\text{Ob}(\mathbf{D}) := \{y\}$ and $\text{End}_{\mathbf{D}}(y) := B$. Then f becomes a \mathbb{K} -linear DG functor $F : \mathbf{C} \rightarrow \mathbf{D}$.

Other examples of DG functors, more relevant to our study, will be given in Subsection 4.4.

Definition 3.5.4. Let $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be \mathbb{K} -linear DG functors.

- (1) A degree i morphism of DG functors $\eta : F \rightarrow G$ is a degree i morphism of graded functors, as in Definition 3.1.17.

- (2) Let $\eta : F \rightarrow G$ be a degree i morphism of DG functors. For any object $M \in \mathbf{C}$ there is a degree $i + 1$ morphism

$$d_D(\eta_M) : F(M) \rightarrow G(M)$$

in \mathbf{D} . We let

$$d_D(\eta) := \{d_D(\eta_M)\}_{M \in \mathbf{C}}.$$

- (3) A *strict morphism of DG functors* is a degree 0 morphism of graded functors $\eta : F \rightarrow G$ such that $d_D(\eta) = 0$.

Proposition 3.5.5. *The collection of morphisms $d_D(\eta)$ defined above is a degree $i + 1$ morphism of DG functors $F \rightarrow G$.*

Proof. Exercise. □

The categories $\text{Str}(\mathbf{C}) = \mathbf{C}_{\text{str}}$ and $\text{Ho}(\mathbf{C})$ were introduced in Definition 3.4.4.

Proposition 3.5.6. *Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a \mathbb{K} -linear DG functor. Then F induces \mathbb{K} -linear functors*

$$\text{Str}(F) : \text{Str}(\mathbf{C}) \rightarrow \text{Str}(\mathbf{D})$$

and

$$\text{Ho}(F) : \text{Ho}(\mathbf{C}) \rightarrow \text{Ho}(\mathbf{D}).$$

Proof. Because F is a DG functor, it sends 0-cocycles in $\text{Hom}_{\mathbf{C}}(M_0, M_1)$ to 0-cocycles in $\text{Hom}_{\mathbf{D}}(F(M_0), F(M_1))$. The same for 0-coboundaries. □

By abuse of notation, and when there is no danger for confusion, we will sometimes write F instead of $\text{Str}(F)$ or $\text{Ho}(F)$.

Exercise 3.5.7. Let \mathbf{A} and \mathbf{C} be \mathbb{K} -linear DG categories, and assume \mathbf{A} is small. Define $\text{DGFun}(\mathbf{A}, \mathbf{C})$ to be the set of \mathbb{K} -linear DG functors $F : \mathbf{A} \rightarrow \mathbf{C}$. Show that $\text{DGFun}(\mathbf{A}, \mathbf{C})$ is a \mathbb{K} -linear DG category, where the morphisms are from Definition 3.5.4(1), and their differentials are from Definition 3.5.4(2).

3.6. Complexes in Abelian Categories. Here we recall facts about complexes from the classical homological theory, and place them within our context. In this subsection \mathbf{M} is a \mathbb{K} -linear abelian category.

Remark 3.6.1. Actually, almost everything we do here makes sense when \mathbf{M} is just a \mathbb{K} -linear category (not necessarily abelian). However, requiring it to be abelian eliminates the confusion between “ring-like” and “module-like” categories (only the latter are abelian). Cf. Remark 3.4.7.

A *complex* of objects of \mathbf{M} , or a complex in \mathbf{M} , is a diagram

$$(3.6.2) \quad (\cdots \rightarrow M^{-1} \xrightarrow{d_M^{-1}} M^0 \xrightarrow{d_M^0} M^1 \xrightarrow{d_M^1} M^2 \rightarrow \cdots)$$

of objects and morphisms in \mathbf{M} , such that $d_M^{i+1} \circ d_M^i = 0$. The collection of objects $M := \{M^i\}_{i \in \mathbb{Z}}$ is nothing but a graded object of \mathbf{M} , as defined in Subsection 3.1. The collection of morphisms $d_M := \{d_M^i\}_{i \in \mathbb{Z}}$ is called a *differential*, or a *coboundary operator*. Thus a complex is a pair (M, d_M) made up of a graded object M and a differential d_M on it. We sometimes write d instead of d_M or d_M^i . At other times we leave the differential implicit, and just refer to the complex as M .

Let N be another complex in \mathbf{M} . A *strict morphism of complexes* $\phi : M \rightarrow N$ is a collection $\phi = \{\phi^i\}_{i \in \mathbb{Z}}$ of morphisms $\phi^i : M^i \rightarrow N^i$ in \mathbf{M} , such that

$$(3.6.3) \quad d_N^i \circ \phi^i = \phi^{i+1} \circ d_M^i.$$

Note that a strict morphism $\phi : M \rightarrow N$ can be viewed as a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M^i & \xrightarrow{d_M^i} & M^{i+1} & \longrightarrow & \cdots \\ & & \downarrow \phi^i & & \downarrow \phi^{i+1} & & \\ \cdots & \longrightarrow & N^i & \xrightarrow{d_N^i} & N^{i+1} & \longrightarrow & \cdots \end{array}$$

in \mathbf{M} . The identity automorphism 1_M of the complex M is a strict morphism.

Remark 3.6.4. In most textbooks, what we call “strict morphism of complexes” is simply called a “morphism of complexes”. See Remark 3.2.3 for an explanation.

Let us denote by $\mathbf{C}_{\text{str}}(\mathbf{M})$ the category of complexes in \mathbf{M} , with strict morphisms. This is a \mathbb{K} -linear abelian category. Indeed, the direct sum of complexes is the degree-wise direct sum, i.e. $(M \oplus N)^i = M^i \oplus N^i$. The same for kernels and cokernels. If \mathbf{N} is a full abelian subcategory of \mathbf{M} , then $\mathbf{C}_{\text{str}}(\mathbf{N})$ is a full abelian subcategory of $\mathbf{C}_{\text{str}}(\mathbf{M})$.

Any single object $M \in \mathbf{M}$ can be viewed as a complex

$$M' := (\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots),$$

where M is in degree 0; the differential of this complex is of course zero. The assignment $M \mapsto M'$ is a fully faithful \mathbb{K} -linear functor $\mathbf{M} \rightarrow \mathbf{C}_{\text{str}}(\mathbf{M})$.

Let M, N be complexes in \mathbf{M} . As in (3.1.19) there is a graded \mathbb{K} -module $\text{Hom}_{\mathbf{M}}(M, N)$. It is a DG \mathbb{K} -module with differential d given by the formula

$$(3.6.5) \quad d(\phi) := d_N \circ \phi - (-1)^i \cdot \phi \circ d_M$$

for $\phi \in \text{Hom}_{\mathbf{M}}(M, N)^i$. It is easy to check that $d \circ d = 0$. We sometimes denote this differential by $d_{\text{Hom}_{\mathbf{M}}(M, N)}$.

Thus, an element $\phi \in \text{Hom}_{\mathbf{M}}(M, N)^i$ is a collection $\phi = \{\phi^j\}_{j \in \mathbb{Z}}$ of morphisms $\phi^j : M^j \rightarrow N^{j+i}$. In a diagram, for $i = 2$, it looks like this:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M^j & \xrightarrow{d} & M^{j+1} & \xrightarrow{d} & M^{j+2} & \xrightarrow{d} & M^{j+3} & \longrightarrow & \cdots \\ & & \searrow \phi^j & & \searrow \phi^{j+1} & & \searrow \phi^{j+2} & & \searrow \phi^{j+3} & & \\ \cdots & \longrightarrow & N^j & \xrightarrow{d} & N^{j+1} & \xrightarrow{d} & N^{j+2} & \xrightarrow{d} & N^{j+3} & \longrightarrow & \cdots \end{array}$$

Since ϕ does not have to commute with the differentials, this is usually not a commutative diagram!

For a triple of complexes M_0, M_1, M_2 , there are \mathbb{K} -linear homomorphisms

$$\text{Hom}_{\mathbf{M}}(M_1, M_2)^{i_2} \otimes_{\mathbb{K}} \text{Hom}_{\mathbf{M}}(M_0, M_1)^{i_1} \rightarrow \text{Hom}_{\mathbf{M}}(M_0, M_2)^{i_1+i_2},$$

$$\phi_2 \otimes \phi_1 \mapsto \phi_2 \circ \phi_1.$$

Lemma 3.6.6. *The composition homomorphism*

$$\text{Hom}_{\mathbf{M}}(M_1, M_2) \otimes_{\mathbb{K}} \text{Hom}_{\mathbf{M}}(M_0, M_1) \rightarrow \text{Hom}_{\mathbf{M}}(M_0, M_2)$$

is a strict homomorphism of DG \mathbb{K} -modules.

Proof. This is a good exercise. \square

The lemma justifies the next definition.

Definition 3.6.7. Let $\mathbf{C}(\mathbf{M})$ be the \mathbb{K} -linear DG category whose objects are the complexes in \mathbf{M} , and the morphism DG \mathbb{K} -modules are $\mathrm{Hom}_{\mathbf{M}}(M, N)$ from formulas (3.1.19) and (3.6.5).

It is clear, from comparing formulas (3.6.5) and (3.6.3), that the strict morphisms of complexes defined at the top of this subsection are the same as those from Definition 3.4.4(1). In other words, $\mathrm{Str}(\mathbf{C}(\mathbf{M})) = \mathbf{C}_{\mathrm{str}}(\mathbf{M})$.

Remark 3.6.8. A possible ambiguity could arise in the meaning of $\mathrm{Hom}_{\mathbf{M}}(M, N)$ if $M, N \in \mathbf{M}$: does it mean the \mathbb{K} -module of morphisms in the category \mathbf{M} ? Or, if we view M and N as complexes by the canonical embedding $\mathbf{M} \subseteq \mathbf{C}(\mathbf{M})$, does $\mathrm{Hom}_{\mathbf{M}}(M, N)$ mean the complex of \mathbb{K} -modules defined for complexes? It turns out that there is no actual difficulty: since the complex of \mathbb{K} -modules $\mathrm{Hom}_{\mathbf{M}}(M, N)$ is concentrated in degree 0, we may view it as a single \mathbb{K} -module, and this is precisely the \mathbb{K} -module of morphisms in the category \mathbf{M} .

When $\mathbf{M} = \mathrm{Mod} A$ for a ring A , there is no essential distinction between complexes and DG modules:

Proposition 3.6.9. *Let A be a central \mathbb{K} -ring. Given a complex $M \in \mathbf{C}(\mathrm{Mod} A)$, with notation as in (3.6.2), define the DG A -module*

$$F(M) := \bigoplus_{i \in \mathbb{Z}} M^i,$$

with differential $d := \sum_{i \in \mathbb{Z}} d_M^i$. Then the functor

$$F : \mathbf{C}(\mathrm{Mod} A) \rightarrow \mathrm{DGMod} A$$

is an equivalence of \mathbb{K} -linear DG categories.

The proof is an exercise. The only hard part in it is to choose good notation.

3.7. The DG Category $\mathbf{C}(A, \mathbf{M})$. We now combine material from previous subsections. The concept introduced in the definition below is new. It is the DG version of Definition 3.1.22.

Definition 3.7.1. Let \mathbf{M} be a \mathbb{K} -linear abelian category, and let A be a central DG \mathbb{K} -ring. A *DG A -module in \mathbf{M}* is an object $M \in \mathbf{C}(\mathbf{M})$, together with a DG \mathbb{K} -ring homomorphism $f : A \rightarrow \mathrm{End}_{\mathbf{M}}(M)$.

If M is a DG A -module in \mathbf{M} , then after forgetting the differentials, M becomes a graded A -module in \mathbf{M} ; see Definition 3.1.22.

Definition 3.7.2. Let \mathbf{M} be a \mathbb{K} -linear abelian category, let A be a central DG \mathbb{K} -ring, and let M, N be DG A -modules in \mathbf{M} . In Definition 3.1.24 we introduced the graded \mathbb{K} -module $\mathrm{Hom}_{A, \mathbf{M}}(M, N)$. This is made into a DG \mathbb{K} -module with differential

$$d(\phi) := d_N \circ \phi - (-1)^i \cdot \phi \circ d_M$$

for $\phi \in \mathrm{Hom}_{A, \mathbf{M}}(M, N)^i$.

When we have to be specific, we denote the differential of $\mathrm{Hom}_{A,M}(M, N)$ by d_{Hom} , $d_{A,M}$, or $d_{\mathrm{Hom}_{A,M}(M,N)}$.

As we have seen before (in Lemma 3.6.6, and Conversely in Lemma 3.4.3), given $\phi_k \in \mathrm{Hom}_{A,M}(M_{k-1}, M_k)^{i_k}$ for $k = 1, 2$, we have

$$\phi_2 \circ \phi_1 \in \mathrm{Hom}_{A,M}(M_0, M_2)^{i_1+i_2},$$

and

$$d(\phi_2 \circ \phi_1) = d(\phi_2) \circ \phi_1 + (-1)^{i_2} \cdot \phi_2 \circ d(\phi_1).$$

Also the identity automorphism $1_M = \mathrm{id}_M$ belongs to $\mathrm{Hom}_{A,M}(M, M)^0$, and $d(1_M) = 0$. Therefore the next definition is legitimate.

Definition 3.7.3. Let M be a \mathbb{K} -linear abelian category, and let A be a central DG \mathbb{K} -ring. The \mathbb{K} -linear DG category of DG A -modules in M is denoted by $\mathbf{C}(A, M)$. The morphism DG module are

$$\mathrm{Hom}_{\mathbf{C}(A, M)}(M_0, M_1) := \mathrm{Hom}_{A, M}(M_0, M_1)$$

from Definition 3.7.2. The composition is that of $\mathbf{C}(M)$.

Notice that forgetting the action of A is a faithful \mathbb{K} -linear DG functor $\mathbf{C}(A, M) \rightarrow \mathbf{C}(M)$. On the other hand, forgetting the differentials is a fully faithful \mathbb{K} -linear graded functor $\mathbf{C}(A, M) \rightarrow \mathbf{G}(A, M)$.

Example 3.7.4. If $A = \mathbb{K}$, then $\mathbf{C}(A, M) = \mathbf{C}(M)$; and if $M = \mathrm{Mod} \mathbb{K}$, then $\mathbf{C}(A, M) = \mathrm{DGMod} A$. Because of this, we sometimes write $\mathbf{C}(A) := \mathrm{DGMod} A$.

Definition 3.7.5. In the situation of Definition 3.7.3:

- (1) The strict category of $\mathbf{C}(A, M)$ (see Definition 3.4.4(1)) is denoted by $\mathbf{C}_{\mathrm{str}}(A, M)$.
- (2) The homotopy category of $\mathbf{C}(A, M)$ (see Definition 3.4.4(2)) is denoted by $\mathbf{K}(A, M)$.

The next proposition is merely an interpretation of the definitions; but it is worth mentioning.

Proposition 3.7.6. *Let $\phi : M \rightarrow N$ be a degree 0 morphism in $\mathbf{C}(A, M)$. The next two conditions are equivalent:*

- (i) ϕ is strict.
- (ii) $\phi \circ d_M = d_N \circ \phi$.

Remark 3.7.7. Here is a generalization of Definition 3.7.3. Instead of a central DG \mathbb{K} -ring A we can take a small \mathbb{K} -linear DG category A . We then define the \mathbb{K} -linear DG category

$$\mathbf{C}(A, M) := \mathrm{DGFun}(A, \mathbf{C}(M))$$

as in Exercise 3.5.7.

This is indeed a generalization of Definition 3.7.3: when A has a single object x , and we write $A := \mathrm{End}_A(x)$, then the functor $M \mapsto M(x)$ is an isomorphism of DG categories $\mathbf{C}(A, M) \xrightarrow{\cong} \mathbf{C}(A, M)$.

In the special case of $M = \mathrm{Mod} \mathbb{K}$, the DG category $\mathbf{C}(A, M)$ is what Keller [Ke] calls the DG category of *left DG A -modules*.

Practically everything we do in this book for $\mathbf{C}(A, M)$ holds in the more general context of $\mathbf{C}(A, M)$. However, in the more general context a lot of the intuition is lost, and some aspects become pretty cumbersome. This is the reason we decided to stick with the less general context.

Definition 3.7.8. Let \mathbf{M} be a \mathbb{K} -linear abelian category, and let A be a central DG \mathbb{K} -ring. For any integer i let

$$H^i : \mathbf{C}_{\text{str}}(A, \mathbf{M}) \rightarrow \mathbf{M}$$

be the \mathbb{K} -linear functor, that sends a complex M to its i -th cohomology $H^i(M) \in \mathbf{M}$ as in (3.2.8), and that sends a strict morphism $\phi : M_0 \rightarrow M_1$ to the morphism

$$H^i(\phi) : H^i(M_0) \rightarrow H^i(M_1).$$

3.8. The Translation Functor. As before, we fix a \mathbb{K} -linear abelian category \mathbf{M} , and a central DG \mathbb{K} -ring A .

The translation functor goes back to the beginnings of derived categories – see Remark 3.8.11. The treatment in this subsection (with the operator t) is taken from [Ye9, Section 1].

Definition 3.8.1. Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a graded module in \mathbf{M} , i.e. an object of $\mathbf{G}(\mathbf{M})$. The *translation* of M is the object

$$T(M) = \{T(M)^i\}_{i \in \mathbb{Z}} \in \mathbf{G}(\mathbf{M})$$

defined as follows: the graded component of degree i of $T(M)$ is $T(M)^i := M^{i+1}$.

Definition 3.8.2 (The little t operator). Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a graded module in \mathbf{M} , i.e. an object of $\mathbf{G}(\mathbf{M})$. We define

$$t_M : M \rightarrow T(M)$$

to be the degree -1 morphism of graded objects of \mathbf{M} , that in every degree $i+1$ is identity morphism

$$t_M := 1_{M^{i+1}} : M^{i+1} \xrightarrow{\cong} M^{i+1} = T(M)^i$$

of the object M^{i+1} in \mathbf{M} .

Note that the morphism

$$t_M \in \text{Hom}_{\mathbf{G}(\mathbf{M})}(M, T(M))^{-1}$$

is invertible, with inverse

$$t_M^{-1} \in \text{Hom}_{\mathbf{G}(\mathbf{M})}(T(M), M)^1.$$

Definition 3.8.3. Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a DG A -module in \mathbf{M} , i.e. an object of $\mathbf{C}(A, \mathbf{M})$. The *translation* of M is the object

$$T(M) \in \mathbf{C}(A, \mathbf{M})$$

defined as follows.

- (1) As graded object of \mathbf{M} , it is as specified in Definition 3.8.1.
- (2) The differential $d_{T(M)}$ is defined by the formula

$$d_{T(M)} := -t_M \circ d_M \circ t_M^{-1}.$$

- (3) Let $f_M : A \rightarrow \text{End}_{\mathbf{M}}(M)$ be the DG ring homomorphism that determines the action of A on M . Then

$$f_{T(M)} : A \rightarrow \text{End}_{\mathbf{M}}(T(M))$$

is defined by

$$f_{T(M)}(a) := (-1)^j \cdot t_M \circ f_M(a) \circ t_M^{-1}$$

for $a \in A^j$.

Thus, the differential $d_{T(M)} = \{d_{T(M)}^i\}_{i \in \mathbb{Z}}$ makes this diagram in \mathbf{M} commutative for every i :

$$\begin{array}{ccc} T(M)^i & \xrightarrow{d_{T(M)}^i} & T(M)^{i+1} \\ \uparrow t_M & & \uparrow t_M \\ M^{i+1} & \xrightarrow{-d_M^{i+1}} & M^{i+2} \end{array}$$

And the left A -module structure makes this diagram in \mathbf{M} commutative for every i and every $a \in A^j$:

$$\begin{array}{ccc} T(M)^i & \xrightarrow{f_{T(M)}(a)} & T(M)^{i+j} \\ \uparrow t_M & & \uparrow t_M \\ M^{i+1} & \xrightarrow{(-1)^j \cdot f_M(a)} & M^{i+j+1} \end{array}$$

Warning: t_M is not a morphism in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, because it has degree -1 .

Proposition 3.8.4. *The morphisms t_M and t_M^{-1} are cocycles, in the DG \mathbb{K} -modules $\text{Hom}_{A, \mathbf{M}}(M, T(M))$ and $\text{Hom}_{A, \mathbf{M}}(T(M), M)$ respectively.*

Proof. We use the notation d_{Hom} for the differential in the DG module $\text{Hom}_{A, \mathbf{M}}(M, T(M))$. Let us calculate. Because t_M has degree -1 , we have

$$\begin{aligned} d_{\text{Hom}}(t_M) &= d_{T(M)} \circ t_M + t_M \circ d_M \\ &= (-t_M \circ d_M \circ t_M^{-1}) \circ t_M + t_M \circ d_M = 0. \end{aligned}$$

As for t_M^{-1} : this is done using the graded Leibniz rule, just like in the proof Proposition 3.4.10. \square

Definition 3.8.5. Given a morphism

$$\phi \in \text{Hom}_{A, \mathbf{M}}(M, N)^i,$$

we define the morphism

$$T(\phi) \in \text{Hom}_{A, \mathbf{M}}(T(M), T(N))^i$$

to be

$$T(\phi) := (-1)^i \cdot t_N \circ \phi \circ t_M^{-1}.$$

To clarify this definition, let us write $\phi = \{\phi^j\}_{j \in \mathbb{Z}}$, so that $\phi^j : M^j \rightarrow N^{j+i}$ is a morphism in \mathbf{M} . Then

$$T(\phi)^j : T(M)^j \rightarrow T(N)^{j+i}$$

is

$$T(\phi)^j = (-1)^i \cdot t_N \circ \phi^{j+1} \circ t_M^{-1}.$$

The corresponding commutative diagram in \mathbf{M} , for each i, j , is:

$$(3.8.6) \quad \begin{array}{ccc} T(M)^j & \xrightarrow{T(\phi)^j} & T(N)^{j+i} \\ \uparrow t_M & & \uparrow t_N \\ M^{j+1} & \xrightarrow{(-1)^i \cdot \phi^{j+1}} & N^{j+i} \end{array}$$

Theorem 3.8.7. *Let \mathbf{M} be \mathbb{K} -linear abelian category and let A be a central DG \mathbb{K} -ring.*

(1) The assignments $M \mapsto T(M)$ and $\phi \mapsto T(\phi)$ are a \mathbb{K} -linear DG functor

$$T : \mathbf{C}(A, M) \rightarrow \mathbf{C}(A, M).$$

(2) The collection $t := \{t_M\}_{M \in \mathbf{C}(A, M)}$ is a degree -1 isomorphism

$$t : \text{Id} \rightarrow T$$

of DG functors from $\mathbf{C}(A, M)$ to itself.

Proof. (1) Take morphisms $\phi_1 : M_0 \rightarrow M_1$ and $\phi_2 : M_1 \rightarrow M_2$, of degrees i_1 and i_2 respectively. Then

$$\begin{aligned} T(\phi_2 \circ \phi_1) &= (-1)^{i_1+i_2} \cdot t_{M_2} \circ (\phi_2 \circ \phi_1) \circ t_{M_0}^{-1} \\ &= (-1)^{i_1+i_2} \cdot t_{M_2} \circ \phi_2 \circ (t_{M_1}^{-1} \circ t_{M_1}) \circ \phi_1 \circ t_{M_0}^{-1} \\ &= ((-1)^{i_2} \cdot t_{M_2} \circ \phi_2 \circ t_{M_1}^{-1}) \circ ((-1)^{i_1} \cdot t_{M_1} \circ \phi_1 \circ t_{M_0}^{-1}) \\ &= T(\phi_2) \circ T(\phi_1). \end{aligned}$$

Clearly $T(1_M) = 1_M$, and

$$T(\lambda \cdot \phi + \psi) = \lambda \cdot T(\phi) + T(\psi)$$

for all $\lambda \in \mathbb{K}$ and $\phi, \psi \in \text{Hom}_{A, M}(M_0, M_1)^i$. So T is a \mathbb{K} -linear graded functor.

By Proposition 3.8.4 we know that $d \circ t = -t \circ d$ and $d \circ t^{-1} = -t^{-1} \circ d$. This implies that for any morphism ϕ in $\mathbf{C}(A, M)$, we have $T(d(\phi)) = d(T(\phi))$. So T is a DG functor.

(2) Take any $\phi \in \text{Hom}_{A, M}(M_0, M_1)^i$. We have to prove that

$$t_{M_1} \circ \phi = (-1)^i \cdot T(\phi) \circ t_{M_0}$$

as elements of $\text{Hom}_{A, M}(M_0, T(M_1))^{i+1}$. But by Definition 3.8.5 we have

$$T(\phi) \circ t_{M_0} = ((-1)^i \cdot t_{M_1} \circ \phi \circ t_{M_0}^{-1}) \circ t_{M_0} = (-1)^i \cdot t_{M_1} \circ \phi.$$

□

Definition 3.8.8. We call T the *translation functor* of the DG category $\mathbf{C}(A, M)$.

Corollary 3.8.9.

- (1) The functor T is an automorphism of the category $\mathbf{C}(A, M)$.
- (2) For any $k, l \in \mathbb{Z}$ there is an equality of functors $T^l \circ T^k = T^{l+k}$.
- (3) For any k the functor

$$T^k : \mathbf{C}(A, M) \rightarrow \mathbf{C}(A, M)$$

is an auto-equivalence of DG categories.

Proof. (1) This is because the functor T is bijective on the set of objects of $\mathbf{C}(A, M)$ and on the sets of morphisms.

(2) By part (1) of this corollary, the inverse T^{-1} is a uniquely defined functor (not just up to an isomorphism of functors).

(3) By part (1) of the theorem above. □

Proposition 3.8.10. Consider any $M \in \mathbf{C}(A, M)$.

- (1) There is equality

$$t_{T(M)} = -T(t_M)$$

of degree -1 morphisms $T(M) \rightarrow T^2(M)$ in $\mathbf{C}(A, M)$.

(2) *There is equality*

$$\mathbf{t}_{T^{-1}(M)} = -T^{-1}(\mathbf{t}_M)$$

of degree -1 morphisms

$$T^{-1}(M) \rightarrow T(T^{-1}(M)) = M = T^{-1}(T(M))$$

in $\mathbf{C}(A, \mathbf{M})$.

Proof. (1) This is an easy calculation, using Definition 3.8.5:

$$T(\mathbf{t}_M) = -\mathbf{t}_{T(M)} \circ \mathbf{t}_M \circ \mathbf{t}_M^{-1} = -\mathbf{t}_{T(M)}.$$

(2) A similar calculation. \square

Remark 3.8.11. There are several names in the literature for the translation functor T : *twist*, *shift* and *suspension*. There are also several notations: $T(M) = M[1] = \Sigma M$. In the later part of this book we shall use the notation $M[k] := T^k(M)$ for the k -th translation.

3.9. The Standard Cone of a Strict Morphism. As before, we fix a \mathbb{K} -linear abelian category \mathbf{M} , and a central DG \mathbb{K} -ring A . Here is the cone construction in $\mathbf{C}(A, \mathbf{M})$, as it looks using the operator \mathbf{t} .

Definition 3.9.1. Let $\phi : M \rightarrow N$ be a strict morphism in $\mathbf{C}(A, \mathbf{M})$. The *standard cone of ϕ* is the object $\text{Cone}(\phi) \in \mathbf{C}(A, \mathbf{M})$ defined as follows. As a graded A -module in \mathbf{M} we let

$$\text{Cone}(\phi) := N \oplus T(M).$$

The differential d_{Cone} is this: if we express the graded module as a column

$$\text{Cone}(\phi) = \begin{bmatrix} N \\ T(M) \end{bmatrix},$$

then d_{Cone} is left multiplication by the matrix

$$d_{\text{Cone}} := \begin{bmatrix} d_N & \phi \circ \mathbf{t}_M^{-1} \\ 0 & d_{T(M)} \end{bmatrix}$$

of degree 1 morphisms of graded A -module in \mathbf{M} .

In other words,

$$d_{\text{Cone}}^i : \text{Cone}(\phi)^i \rightarrow \text{Cone}(\phi)^{i+1}$$

is

$$d_{\text{Cone}}^i = d_N^i + d_{T(M)}^i + \phi^{i+1} \circ \mathbf{t}_M^{-1},$$

where $\phi^{i+1} \circ \mathbf{t}_M^{-1}$ is the composed morphism

$$T(M)^i \xrightarrow{\mathbf{t}_M^{-1}} M^{i+1} \xrightarrow{\phi^{i+1}} N^{i+1}.$$

Let us denote by

$$(3.9.2) \quad e_\phi : N \rightarrow N \oplus T(M)$$

the embedding, and by

$$(3.9.3) \quad p_\phi : N \oplus T(M) \rightarrow T(M)$$

the projection. Thus, as matrices we have

$$e_\phi = \begin{bmatrix} 1_N \\ 0 \end{bmatrix} \quad \text{and} \quad p_\phi = \begin{bmatrix} 0 & 1_{T(M)} \end{bmatrix}.$$

The standard cone of ϕ sits in the exact sequence

$$(3.9.4) \quad 0 \rightarrow N \xrightarrow{e_\phi} \text{Cone}(\phi) \xrightarrow{p_\phi} T(M) \rightarrow 0$$

in the abelian category $\mathbf{C}_{\text{str}}(A, \mathbf{M})$.

Definition 3.9.5. Let $\phi : M \rightarrow N$ be a morphism in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. The diagram

$$M \xrightarrow{\phi} N \xrightarrow{e_\phi} \text{Cone}(\phi) \xrightarrow{p_\phi} T(M)$$

in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ is called the *standard triangle* associated to ϕ .

The cone construction is functorial, in the following sense.

Proposition 3.9.6. *Let*

$$\begin{array}{ccc} M_0 & \xrightarrow{\phi_0} & N_0 \\ \psi \downarrow & & \downarrow \chi \\ M_1 & \xrightarrow{\phi_1} & N_1 \end{array}$$

be a commutative diagram in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. Then

$$(3.9.7) \quad (\chi, T(\psi)) : \text{Cone}(\phi_0) \rightarrow \text{Cone}(\phi_1)$$

is a morphism in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, and the diagram

$$\begin{array}{ccccccc} M_0 & \xrightarrow{\phi_0} & N_0 & \xrightarrow{e_{\phi_0}} & \text{Cone}(\phi_0) & \xrightarrow{p_{\phi_0}} & T(M_0) \\ \psi \downarrow & & \downarrow \chi & & \downarrow (\chi, T(\psi)) & & \downarrow T(\psi) \\ M_1 & \xrightarrow{\phi_1} & N_1 & \xrightarrow{e_{\phi_1}} & \text{Cone}(\phi_1) & \xrightarrow{p_{\phi_1}} & T(M_1) \end{array}$$

in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ is commutative.

Proof. This is a simple consequence of the definitions. □

4. PROPERTIES OF DG FUNCTORS

In this section we fix central DG \mathbb{K} -rings A and B , and \mathbb{K} -linear abelian categories \mathbf{M} and \mathbf{N} . The \mathbb{K} -linear DG categories $\mathbf{C}(A, \mathbf{M})$ and $\mathbf{C}(B, \mathbf{N})$ were introduced in Subsection 3.7. Graded functors, DG functors, and morphisms between them, were introduced in Subsection 3.5.

Some of the material in this section is new – such as the gauge in Definition 4.1.1, and its role in the characterization of DG functors in Theorem 4.1.2. We think that Theorem 4.3.7, which says that a DG functor commutes with cones, is also a new result.

4.1. The Gauge of a Graded Functor. The next definition is new.

Definition 4.1.1. Let

$$F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N})$$

be a \mathbb{K} -linear graded functor. For any object $M \in \mathbf{C}(A, \mathbf{M})$ let

$$\gamma_{F,M} := d_{F(M)} - F(d_M) \in \text{Hom}_{B,\mathbf{N}}(F(M), F(M))^1.$$

The collection of morphisms

$$\gamma_F := \{\gamma_{F,M}\}_{M \in \mathbf{C}(A, \mathbf{M})}$$

is called the *gauge* of F .

Theorem 4.1.2.¹ *The following two conditions are equivalent for a \mathbb{K} -linear graded functor*

$$F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N}).$$

- (i) *F is a DG functor.*
- (ii) *The gauge γ_F is a degree 1 morphism of graded functors $\gamma_F : F \rightarrow F$.*

Proof. Recall that F is a DG functor (condition (i)) iff

$$(4.1.3) \quad (F \circ d_{A,\mathbf{M}})(\phi) = (d_{B,\mathbf{N}} \circ F)(\phi)$$

for every $\phi \in \text{Hom}_{A,\mathbf{M}}(M_0, M_1)^i$. And γ_F is a degree 1 morphism of graded functors (condition (ii)) iff

$$(4.1.4) \quad \gamma_{F,M_1} \circ F(\phi) = (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0}$$

for every such ϕ .

Here is the calculation. Because F is a graded functor, we get

$$(4.1.5) \quad \begin{aligned} F(d_{A,\mathbf{M}}(\phi)) &= F(d_{M_1} \circ \phi - (-1)^i \cdot \phi \circ d_{M_0}) \\ &= F(d_{M_1}) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ F(d_{M_0}) \end{aligned}$$

and

$$(4.1.6) \quad d_{B,\mathbf{N}}(F(\phi)) = d_{F(M_1)} \circ F(\phi) - (-1)^i \cdot F(\phi) \circ d_{F(M_0)}.$$

Using equations (4.1.5) and (4.1.6), and the definition of γ_F , we obtain

$$(4.1.7) \quad \begin{aligned} (F \circ d_{A,\mathbf{M}} - d_{B,\mathbf{N}} \circ F)(\phi) &= F(d_{A,\mathbf{M}}(\phi)) - d_{B,\mathbf{N}}(F(\phi)) \\ &= (F(d_{M_1}) - d_{F(M_1)}) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ (F(d_{M_0}) - d_{F(M_0)}) \\ &= -\gamma_{F,M_1} \circ F(\phi) + (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0}. \end{aligned}$$

¹This theorem is due to Rishi Vyas.

Finally, the vanishing of the first expression in (4.1.7) is the same as equality in (4.1.3); whereas the vanishing of the last expression in (4.1.7) is the same as equality in (4.1.4). \square

4.2. The Translation Isomorphism of a DG Functor. The translation functor of $\mathbf{C}(A, M)$ will be denoted here by $T_{A,M}$. Recall that for an object $M \in \mathbf{C}(A, M)$, we have the little t operator

$$t_M \in \text{Hom}_{A,M}(M, T_{A,M}(M))^{-1}.$$

This is an isomorphism in $\mathbf{C}(A, M)$. Likewise for the DG category $\mathbf{C}(B, N)$.

Definition 4.2.1. Let

$$F : \mathbf{C}(A, M) \rightarrow \mathbf{C}(B, N)$$

be a \mathbb{K} -linear DG functor. For an object $M \in \mathbf{C}(A, M)$, let

$$\tau_{F,M} : F(T_{A,M}(M)) \rightarrow T_{B,N}(F(M))$$

be the isomorphism

$$\tau_{F,M} := t_{F(M)} \circ F(t_M)^{-1}$$

in $\mathbf{C}(B, N)$, called the *translation isomorphism* of the functor F at the object M .

The isomorphism $\tau_{F,M}$ sits in the following commutative diagram

$$\begin{array}{ccc} F(T_{A,M}(M)) & \xrightarrow{\tau_{F,M}} & T_{B,N}(F(M)) \\ \uparrow F(t_M) & \nearrow t_{F(M)} & \\ F(M) & & \end{array}$$

of isomorphisms in the category $\mathbf{C}(B, N)$.

Proposition 4.2.2. $\tau_{F,M}$ is an isomorphism in $\mathbf{C}_{\text{str}}(B, N)$.

Proof. We know that $\tau_{F,M}$ is an isomorphism in $\mathbf{C}(B, N)$. It suffices to prove that both $\tau_{F,M}$ and its inverse $\tau_{F,M}^{-1}$ are strict morphisms. Now by Proposition 3.8.4, t_M and t_M^{-1} are cocycles. Therefore, $F(t_M)$ and $F(t_M)^{-1} = F(t_M^{-1})$ are cocycles. For the same reason, $t_{F(M)}$ and $t_{F(M)}^{-1}$ are cocycles. But $\tau_{F,M} = t_{F(M)} \circ F(t_M)^{-1}$, and $\tau_{F,M}^{-1} = F(t_M) \circ t_{F(M)}^{-1}$. \square

Theorem 4.2.3. Let

$$F : \mathbf{C}(A, M) \rightarrow \mathbf{C}(B, N)$$

be a \mathbb{K} -linear DG functor. Then the collection $\tau_F := \{\tau_{F,M}\}_{M \in \mathbf{C}(A, M)}$ is an isomorphism

$$\tau_F : F \circ T_{A,M} \xrightarrow{\cong} T_{B,N} \circ F$$

of functors

$$\mathbf{C}_{\text{str}}(A, M) \rightarrow \mathbf{C}_{\text{str}}(B, N).$$

The slogan summarizing this theorem is “A DG functor commutes with translations”.

Proof. In view of Proposition 4.2.2, all we need to prove is that τ_F is a morphism of functors (i.e. it is a natural transformation).

Let $\phi : M_0 \rightarrow M_1$ be a morphism in $\mathbf{C}_{\text{str}}(A, M)$. We must prove that the diagram

$$\begin{array}{ccc} (F \circ T_{A,M})(M_0) & \xrightarrow{\tau_{F,M_0}} & (T_{B,N} \circ F)(M_0) \\ (F \circ T_{A,M})(\phi) \downarrow & & \downarrow (T_{B,N} \circ F)(\phi) \\ (F \circ T_{A,M})(M_1) & \xrightarrow{\tau_{F,M_1}} & (T_{B,N} \circ F)(M_1) \end{array}$$

in $\mathbf{C}_{\text{str}}(B, N)$ is commutative. This will be true if the next diagram

$$\begin{array}{ccccc} (F \circ T_{A,M})(M_0) & \xleftarrow{F(t_{M_0})} & F(M_0) & \xrightarrow{t_{F(M_0)}} & (T_{B,N} \circ F)(M_0) \\ (F \circ T_{A,M})(\phi) \downarrow & & F(\phi) \downarrow & & \downarrow (T_{B,N} \circ F)(\phi) \\ (F \circ T_{A,M})(M_1) & \xleftarrow{F(t_{M_1})} & F(M_1) & \xrightarrow{t_{F(M_1)}} & (T_{B,N} \circ F)(M_1) \end{array}$$

in $\mathbf{C}(B, N)$, whose horizontal arrows are isomorphisms, is commutative. For this to be true, it is enough to prove that both squares in this diagram are commutative. This is true by Theorem 3.8.7(2) \square

Recall that the translation T and all its powers are DG functors. To finish this subsection, we calculate their translation isomorphisms.

Proposition 4.2.4. *For any integer k , the translation isomorphism of the DG functor T^k is*

$$\tau_{T^k} = (-1)^k \cdot \text{id}_{T^{k+1}},$$

where $\text{id}_{T^{k+1}}$ is the identity automorphism of the functor T^{k+1} .

Proof. By Definition 4.2.1 and Proposition 3.8.10(1), for $k = 1$ the formula is

$$\tau_{T,M} = t_{T(M)} \circ T(t_M)^{-1} = -\text{id}_{T^2(M)},$$

where $\text{id}_{T^2(M)}$ is the identity automorphism of the DG module $T^2(M)$. Hence $\tau_T = -\text{id}_{T^2}$. For other integers k the calculation is similar. \square

4.3. Cones and DG Functors.

Definition 4.3.1. The subcategory $\mathbf{C}^0(A, M)$ of $\mathbf{C}(A, M)$ is defined to be the subcategory on all objects, but with degree 0 morphisms only.

There are inclusions of categories (faithful functors, identities on objects)

$$\mathbf{C}_{\text{str}}(A, M) \xrightarrow{\subseteq} \mathbf{C}^0(A, M) \xrightarrow{\subseteq} \mathbf{C}(A, M).$$

Forgetting the differentials is a fully faithful functor

$$(4.3.2) \quad \mathbf{C}^0(A, M) \rightarrow \mathbf{G}^0(A, M);$$

see Definition 3.1.13.

Let

$$F : \mathbf{C}(A, M) \rightarrow \mathbf{C}(B, N)$$

be a \mathbb{K} -linear DG functor. Given a morphism $\phi : M_0 \rightarrow M_1$ in $\mathbf{C}_{\text{str}}(A, M)$, we have a morphism

$$F(\phi) : F(M_0) \rightarrow F(M_1)$$

in $\mathbf{C}_{\text{str}}(B, \mathbf{N})$, and objects $F(\text{Cone}_{A, \mathbf{M}}(\phi))$ and $\text{Cone}_{B, \mathbf{N}}(F(\phi))$ in $\mathbf{C}(B, \mathbf{N})$. By definition (and the fully faithful functor (4.3.2)) there is a canonical isomorphism

$$(4.3.3) \quad \text{Cone}_{A, \mathbf{M}}(\phi) \cong M_1 \oplus T_{A, \mathbf{M}}(M_0)$$

in $\mathbf{C}^0(A, \mathbf{M})$. Since F is an additive functor, it commutes with finite direct sums, and therefore there is a canonical isomorphism

$$(4.3.4) \quad F(\text{Cone}_{A, \mathbf{M}}(\phi)) \cong F(M_1) \oplus F(T_{A, \mathbf{M}}(M_0))$$

in $\mathbf{C}^0(B, \mathbf{N})$. And by definition there is a canonical isomorphism

$$(4.3.5) \quad \text{Cone}_{B, \mathbf{N}}(F(\phi)) \cong F(M_1) \oplus T_{B, \mathbf{N}}(F(M_0))$$

in $\mathbf{C}^0(B, \mathbf{N})$. Warning: the isomorphisms (4.3.3), (4.3.4) and (4.3.5) are usually not strict! They are degree 0 isomorphisms of graded modules, but they might not commute with the differentials; see Proposition 3.7.6. The differentials on the right sides are diagonal matrices, but on the left sides they are upper-triangular matrices (see Definition 3.9.1).

Lemma 4.3.6. *Let*

$$F, G : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N})$$

be \mathbb{K} -linear graded functors, and let $\eta : F \rightarrow G$ be a degree j morphism of graded functors. Suppose $M \cong M_0 \oplus M_1$ in $\mathbf{C}^0(A, \mathbf{M})$, with embeddings $e_i : M_i \rightarrow M$ and projections $p_i : M \rightarrow M_i$. Then

$$\eta_M = (G(e_0), G(e_1)) \circ (\eta_{M_0}, \eta_{M_1}) \circ (F(p_0), F(p_1)),$$

as degree j morphisms $F(M) \rightarrow G(M)$ in $\mathbf{C}(B, \mathbf{N})$.

The lemma says that the diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{(F(p_0), F(p_1))} & F(M_0) \oplus F(M_1) \\ \eta_M \downarrow & & \downarrow (\eta_{M_0}, \eta_{M_1}) \\ G(M) & \xleftarrow{(G(e_0), G(e_1))} & G(M_0) \oplus G(M_1) \end{array}$$

in $\mathbf{C}(B, \mathbf{N})$ is commutative.

Proof. It suffices to prove that the diagram below is commutative for $i = 0, 1$:

$$\begin{array}{ccccc} & & \text{id} & & \\ & \nearrow & & \searrow & \\ F(M_i) & \xrightarrow{F(e_i)} & F(M) & \xrightarrow{F(p_i)} & F(M_i) \\ \eta_{M_i} \downarrow & & \eta_M \downarrow & & \eta_{M_i} \downarrow \\ G(M_i) & \xrightarrow{G(e_i)} & G(M) & \xrightarrow{G(p_i)} & G(M_i) \\ & \nwarrow & & \nearrow & \\ & & \text{id} & & \end{array}$$

This is true because η is a morphism of functors (a natural transformation). \square

Theorem 4.3.7. *Let*

$$F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N})$$

be a \mathbb{K} -linear DG functor, and let $\phi : M_0 \rightarrow M_1$ be a morphism in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. Define the isomorphism

$$\text{cone}(F, \phi) : F(\text{Cone}_{A, \mathbf{M}}(\phi)) \rightarrow \text{Cone}_{B, \mathbf{N}}(F(\phi))$$

in $\mathbf{C}^0(B, \mathbf{N})$ to be

$$\text{cone}(F, \phi) := (\text{id}_{F(M_1)}, \tau_{F, M_0}).$$

Then:

- (1) *The isomorphism $\text{cone}(F, \phi)$ is strict; namely it commutes with the differentials.*
- (2) *The diagram*

$$\begin{array}{ccccccc} F(M_0) & \xrightarrow{F(\phi)} & F(M_1) & \xrightarrow{F(e_\phi)} & F(\text{Cone}_{A, \mathbf{M}}(\phi)) & \xrightarrow{F(p_\phi)} & F(\text{T}_{A, \mathbf{M}}(M_0)) \\ \downarrow = & & \downarrow = & & \downarrow \text{cone}(F, \phi) & & \downarrow \tau_{F, M_0} \\ F(M_0) & \xrightarrow{F(\phi)} & F(M_1) & \xrightarrow{e_{F(\phi)}} & \text{Cone}_{B, \mathbf{N}}(F(\phi)) & \xrightarrow{p_{F(\phi)}} & \text{T}_{B, \mathbf{N}}(F(M_0)) \end{array}$$

in $\mathbf{C}_{\text{str}}(B, \mathbf{N})$ is commutative.

When defining $\text{cone}(F, \phi)$ above, we are using the decompositions (4.3.4) and (4.3.5) in the category $\mathbf{C}^0(B, \mathbf{N})$, and the isomorphism τ_{F, M_0} from Definition 4.2.1.

The slogan summarizing this theorem is “A DG functor sends standard triangles to standard triangles”.

Proof. (1) To save space let us write $\theta := \text{cone}(F, \phi)$. We have to prove that $d_{B, \mathbf{N}}(\theta) = 0$. Let's write $P := \text{Cone}_{A, \mathbf{M}}(\phi)$ and $Q := \text{Cone}_{B, \mathbf{N}}(F(\phi))$. Recall that

$$d_{B, \mathbf{N}}(\theta) = d_Q \circ \theta - \theta \circ d_{F(P)}.$$

We have to prove that this is the zero element in $\text{Hom}_{B, \mathbf{N}}(F(P), Q)^1$.

Writing the cones as column modules:

$$P = \begin{bmatrix} M_1 \\ \text{T}_{A, \mathbf{M}}(M_0) \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} F(M_1) \\ \text{T}_{B, \mathbf{N}}(F(M_0)) \end{bmatrix},$$

the matrices representing the morphisms in question are

$$\theta = \begin{bmatrix} \text{id}_{F(M_1)} & 0 \\ 0 & \tau_{F, M_0} \end{bmatrix}, \quad d_P = \begin{bmatrix} d_{M_1} & \phi \circ t_{M_0}^{-1} \\ 0 & d_{\text{T}_{A, \mathbf{M}}(M_0)} \end{bmatrix}$$

and

$$d_Q = \begin{bmatrix} d_{F(M_1)} & F(\phi) \circ t_{F(M_0)}^{-1} \\ 0 & d_{\text{T}_{B, \mathbf{N}}(F(M_0))} \end{bmatrix}.$$

Let us write $\gamma := \gamma_F$ for simplicity. According to Theorem 4.1.2, the gauge $\gamma : F \rightarrow F$ is a degree 1 morphism of functors $\mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N})$. Because the decomposition (4.3.3) is in the category $\mathbf{C}^0(A, \mathbf{M})$, Lemma 4.3.6 tells us that γ_P decomposes too, i.e.

$$\gamma_P = \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{\text{T}_{A, \mathbf{M}}(M_0)} \end{bmatrix}.$$

By definition of γ_P we have

$$d_{F(P)} = F(d_P) + \gamma_P \in \text{Hom}_{B, \mathbf{N}}(F(P), F(P))^1.$$

It follows that

$$\begin{aligned} d_{F(P)} &= F(d_P) + \gamma_P \\ &= \begin{bmatrix} F(d_{M_1}) & F(\phi \circ t_{M_0}^{-1}) \\ 0 & F(d_{T_{A, \mathbf{M}}(M_0)}) \end{bmatrix} + \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{T_{A, \mathbf{M}}(M_0)} \end{bmatrix} \\ &= \begin{bmatrix} F(d_{M_1}) + \gamma_{M_1} & F(\phi \circ t_{M_0}^{-1}) \\ 0 & F(d_{T_{A, \mathbf{M}}(M_0)}) + \gamma_{T_{A, \mathbf{M}}(M_0)} \end{bmatrix} \\ &= \begin{bmatrix} d_{F(M_1)} & F(\phi \circ t_{M_0}^{-1}) \\ 0 & d_{F(T_{A, \mathbf{M}}(M_0))} \end{bmatrix}. \end{aligned}$$

Finally we will check that $\theta \circ d_{F(P)}$ and $d_Q \circ \theta$ are equal as matrices of morphisms. We do that in each matrix position separately. The two left positions in the matrices $\theta \circ d_{F(P)}$ and $d_Q \circ \theta$ agree trivially. The bottom right positions in these matrices are $\tau_{F, M_0} \circ d_{F(T_{A, \mathbf{M}}(M_0))}$ and $d_{T_{B, \mathbf{N}}(F(M_0))} \circ \tau_{F, M_0}$ respectively; they are equal by Proposition 4.2.2. And in the top right positions we have $F(\phi \circ t_{M_0}^{-1})$ and $F(\phi) \circ t_{F(M_0)}^{-1} \circ \tau_{F, M_0}$ respectively. Now $F(\phi \circ t_{M_0}^{-1}) = F(\phi) \circ F(t_{M_0}^{-1})$; so it suffices to prove that $F(t_{M_0}^{-1}) = t_{F(M_0)}^{-1} \circ \tau_{F, M_0}$. This is immediate from the definition of τ_{F, M_0} .

(2) By definition of $\theta = \text{cone}(F, \phi)$, the diagram is commutative in $\mathbf{C}^0(B, \mathbf{N})$. But by part (1) we know that all morphisms in it lie in $\mathbf{C}_{\text{str}}(B, \mathbf{N})$. \square

Corollary 4.3.8. *In the situation of Theorem 4.3.7, the diagram*

$$\begin{array}{ccccccc} F(M_0) & \xrightarrow{F(\phi)} & F(M_1) & \xrightarrow{F(e_\phi)} & F(\text{Cone}_{A, \mathbf{M}}(\phi)) & \xrightarrow{\tau_{F, M_0} \circ F(p_\phi)} & T_{B, \mathbf{N}}(F(M_0)) \\ \downarrow = & & \downarrow = & & \downarrow \text{cone}(F, \phi) & & \downarrow = \\ F(M_0) & \xrightarrow{F(\phi)} & F(M_1) & \xrightarrow{e_{F(\phi)}} & \text{Cone}_{B, \mathbf{N}}(F(\phi)) & \xrightarrow{p_{F(\phi)}} & T_{B, \mathbf{N}}(F(M_0)) \end{array}$$

is an isomorphism of triangles in $\mathbf{C}_{\text{str}}(B, \mathbf{N})$.

Proof. Just rearrange the diagram in item (2) of the theorem. \square

4.4. Examples of DG Functors. Recall that \mathbf{M} and \mathbf{N} are \mathbb{K} -linear categories, and A and B are central DG \mathbb{K} -rings. Here are three examples of DG functors, of various types. These examples should serve as templates for constructing other DG functors.

Example 4.4.1. Here $A = B = \mathbb{K}$, so $\mathbf{C}(A, \mathbf{M}) = \mathbf{C}(\mathbf{M})$ and $\mathbf{C}(B, \mathbf{N}) = \mathbf{C}(\mathbf{N})$. Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be a \mathbb{K} -linear functor. It extends to a functor

$$\mathbf{C}(F) : \mathbf{C}(\mathbf{M}) \rightarrow \mathbf{C}(\mathbf{N})$$

as follows: on objects, a complex

$$M = (\{M^i\}_{i \in \mathbb{Z}}, \{d_M^i\}_{i \in \mathbb{Z}}) \in \mathbf{C}(\mathbf{M})$$

goes to the complex

$$\mathbf{C}(F)(M) := (\{F(M^i)\}, \{F(d_M^i)\}) \in \mathbf{C}(\mathbf{N}).$$

A morphism $\phi = \{\phi^j\}$ in $\mathbf{C}(\mathbf{M})$ goes to the morphism $\mathbf{C}(\phi) := \{F(\phi^j)\}$ in $\mathbf{C}(\mathbf{N})$. A slightly tedious calculation shows that $\mathbf{C}(F)$ is a \mathbb{K} -linear DG functor.

Given a complex $M \in \mathbf{C}(\mathbf{M})$, let $N := \mathbf{C}(F)(M) \in \mathbf{C}(\mathbf{N})$. Then the translations are

$$\mathrm{T}_{\mathbf{N}}(N) = \mathbf{C}(F)(\mathrm{T}_{\mathbf{M}}(M));$$

and $\mathbf{C}(F)(t_M) = t_N$. So the translation isomorphism

$$\tau_{\mathbf{C}(F)} : \mathbf{C}(F) \circ \mathrm{T}_{\mathbf{M}} \xrightarrow{\sim} \mathrm{T}_{\mathbf{N}} \circ \mathbf{C}(F)$$

of functors $\mathbf{C}_{\mathrm{str}}(\mathbf{M}) \rightarrow \mathbf{C}_{\mathrm{str}}(\mathbf{N})$ is equality.

Let $\phi : M_0 \rightarrow M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(\mathbf{M})$, whose image under $\mathbf{C}(F)$ is the morphism $\psi : N_0 \rightarrow N_1$ in $\mathbf{C}_{\mathrm{str}}(\mathbf{N})$. Then

$$\mathrm{Cone}(\psi) = N_1 \oplus \mathrm{T}_{\mathbf{N}}(N_0) = \mathbf{C}(F)(\mathrm{Cone}(\phi))$$

as graded objects of \mathbf{N} , with differential

$$d_{\mathrm{Cone}(\psi)} = \begin{bmatrix} d_{N_1} & \psi \circ t_{N_0}^{-1} \\ 0 & d_{\mathrm{T}(N_0)} \end{bmatrix} = \mathbf{C}(F) \left(\begin{bmatrix} d_{M_1} & \phi \circ t_{M_0}^{-1} \\ 0 & d_{\mathrm{T}(M_0)} \end{bmatrix} \right) = \mathbf{C}(F)(d_{\mathrm{Cone}(\phi)}).$$

We see that the cone isomorphism θ is equality, and the gauge $\gamma_{\mathbf{C}(F)}$ is zero.

The next example is much more complicated, and we work out the full details (only once – later on, such details will be left to the reader).

Example 4.4.2. Let A and B be central DG \mathbb{K} -rings, and fix some

$$N \in \mathrm{DGMod}(B \otimes_{\mathbb{K}} A^{\mathrm{op}}).$$

In other words, N is a DG B - A -bimodule. For any $M \in \mathrm{DGMod} A$ we have a DG \mathbb{K} -module

$$F(M) := N \otimes_A M,$$

as in Definition 3.3.19. The differential of $F(M)$ is

$$(4.4.3) \quad d_{F(M)} = d_N \otimes \mathrm{id}_M + \mathrm{id}_N \otimes d_M.$$

See Example 3.1.4 regarding the Koszul sign rule that's involved. But $F(M)$ has a structure of a DG B -module: for any $b \in B$, $n \in N$ and $m \in M$, the action is

$$b \cdot (n \otimes m) := (b \cdot n) \otimes m.$$

Clearly

$$F : \mathbf{C}(A) = \mathrm{DGMod} A \rightarrow \mathbf{C}(B) = \mathrm{DGMod} B$$

is a \mathbb{K} -linear functor. We will show that it is actually a DG functor.

Let $M_0, M_1 \in \mathbf{C}(A)$, and consider the \mathbb{K} -linear homomorphism

$$(4.4.4) \quad F : \mathrm{Hom}_A(M_0, M_1) \rightarrow \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1).$$

Take any $\phi \in \mathrm{Hom}_A(M_0, M_1)^i$. Then

$$F(\phi) \in \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1)$$

is the homomorphism that on a homogeneous tensor $n \otimes m \in (N \otimes_A M_0)^{k+j}$, with $n \in N^k$ and $m \in M_0^j$, has the value

$$F(\phi)(n \otimes m) = (-1)^{ik} \cdot n \otimes \phi(m) \in (N \otimes_A M_1)^{k+j+i}.$$

In other words,

$$(4.4.5) \quad F(\phi) = \mathrm{id}_N \otimes \phi.$$

We see that the homomorphism $F(\phi)$ has degree i . So F is a graded functor.

Let us calculate γ_F , the gauge of F . From (4.4.5) and (4.4.3) we get

$$\gamma_{F,M} = d_N \otimes \text{id}_M,$$

which is often a nonzero endomorphism of $F(M)$. Still, take any degree i morphism $\phi : M_0 \rightarrow M_1$ in $\mathbf{C}(A)$. Then

$$\begin{aligned} \gamma_{M_1} \circ F(\phi) &= (d_N \otimes \text{id}_{M_1}) \circ (\text{id}_N \otimes \phi) \\ &= d_N \otimes \phi = (-1)^i \cdot (\text{id}_N \otimes \phi) \circ (d_N \otimes \text{id}_{M_0}) = (-1)^i \cdot F(\phi) \circ \gamma_{M_0}. \end{aligned}$$

We see that γ_F satisfies the condition of Definition 3.5.4(1), which is really Definition 3.1.17. By Theorem 4.1.2, F is a DG functor. (It is possible to calculate directly that F is a DG functor, but this takes more work.)

Finally let us figure out what is the translation isomorphism τ_F of the functor F . Take $M \in \mathbf{C}(A)$. Then

$$\tau_{F,M} : F(T_A(M)) \rightarrow T_B(F(M))$$

is an isomorphism in $\mathbf{C}_{\text{str}}(B)$. By Definition 4.2.1 we have $\tau_{F,M} := t_{F(M)} \circ F(t_M)^{-1}$. Take any $n \in N^k$ and $m \in M^{j+1}$, so that

$$n \otimes t_M(m) \in (N \otimes_A T_A(M))^{k+j} = F(T_A(M))^{k+j},$$

a typical degree $k+j$ element of $F(T_A(M))$. But

$$n \otimes t_M(m) = (-1)^k \cdot (\text{id}_N \otimes t_M)(n \otimes m) = (-1)^k \cdot F(t_M)(n \otimes m).$$

Therefore

$$\tau_{F,M}(n \otimes t_M(m)) = (-1)^k \cdot t_{F(M)}(n \otimes m) \in T_B(F(M))^{k+j}.$$

Observe that when N is concentrated in degree 0, we are back in the situation of Example 4.4.1, in which there are no sign twists, and $\tau_{F,M}$ is “equality”.

Example 4.4.6. Let A and B be central DG \mathbb{K} -rings, and fix some

$$N \in \text{DGM}od(A \otimes_{\mathbb{K}} B^{\text{op}}).$$

For any $M \in \text{DGM}od A$ we define

$$F(M) := \text{Hom}_A(N, M).$$

This is a DG B -module: for any $b \in B^i$ and $\phi \in \text{Hom}_A(N, M)^j$, the homomorphism $b \cdot \phi \in \text{Hom}_A(N, M)^{i+j}$ has value

$$(b \cdot \phi)(n) := (-1)^{i \cdot (j+k)} \cdot \phi(n \cdot b) \in M^{i+j+k}$$

on $n \in N^k$. As in the previous example,

$$F : \mathbf{C}(A) = \text{DGM}od A \rightarrow \mathbf{C}(B) = \text{DGM}od B$$

is a \mathbb{K} -linear graded functor.

The value of the gauge γ_F at $M \in \mathbf{C}(A)$ is

$$\gamma_{F,M} = \text{Hom}(d_N, \text{id}_M).$$

See Example 3.1.5 regarding this notation. Namely for

$$\psi \in F(M)^j = \text{Hom}_A(N, M)^j$$

we have

$$\gamma_{F,M}(\psi) = (-1)^j \cdot \psi \circ d_N.$$

It is not too hard to check that γ_F is a degree 1 morphism of functors. Hence, by Theorem 4.1.2, F is a DG functor.

The formula for the translation isomorphism τ_F is as follows. Take $M \in \mathbf{C}(A)$. Then

$$\tau_{F,M} : F(T_A(M)) = \mathrm{Hom}_A(N, T_A(M)) \rightarrow T_B(F(M)) = T_B(\mathrm{Hom}_A(N, M))$$

is, by definition, $\tau_{F,M} = t_{F(M)} \circ F(t_M)^{-1}$. Now

$$F(t_M)^{-1} = \mathrm{Hom}(\mathrm{id}_N, t_M^{-1}).$$

So given any $\psi \in F(T_A(M))^k$, we have

$$\tau_{F,M}(\psi) = t_{F(M)}(t_M^{-1} \circ \psi) \in T_B(F(M))^k.$$

5. PRETRIANGULATED CATEGORIES AND TRIANGULATED FUNCTORS

In this section we introduce pretriangulated categories and triangulated functors. “Pretriangulated” means that the octahedral axiom is not required to hold. There is one result here that seems to be new: Theorem 5.4.13, which asserts that a DG functor between DG module categories induces a triangulated functor between the associated homotopy categories.

As in previous sections, we fix a base commutative ring \mathbb{K} . All linear categories and linear functors here are implicitly assumed to be \mathbb{K} -linear. In particular, this assumption says that all DG rings are central \mathbb{K} -rings, and all DG ring homomorphisms are \mathbb{K} -linear.

5.1. T-Additive Categories. Recall that a functor is called an isomorphism of categories if it is bijective of sets of objects and on sets of morphisms; see Example 1.5.2.

Definition 5.1.1. Let \mathcal{K} be an additive category. A *translation* on \mathcal{K} is an additive automorphism T of \mathcal{K} , called the *translation functor*. The pair (\mathcal{K}, T) is called a *T-additive category*.

Remark 5.1.2. Some texts give a more relaxed definition: T is only required to be an additive auto-equivalence of \mathcal{K} . The resulting theory is more complicated (it is 2-categorical, but most texts try to suppress this fact).

Later in the book we will write $M[k] := T^k(M)$, the k -th translation of an object M .

Definition 5.1.3. Suppose $(\mathcal{K}, T_{\mathcal{K}})$ and $(\mathcal{L}, T_{\mathcal{L}})$ are T-additive categories. A *T-additive functor* between them is a pair (F, τ) , consisting of an additive functor $F : \mathcal{K} \rightarrow \mathcal{L}$, together with an isomorphism

$$\tau : F \circ T_{\mathcal{K}} \xrightarrow{\sim} T_{\mathcal{L}} \circ F$$

of functors $\mathcal{K} \rightarrow \mathcal{L}$, called a *translation isomorphism*.

Definition 5.1.4. Let (\mathcal{K}_i, T_i) be T-additive categories, for $i = 0, 1, 2$, and let

$$(F_i, \tau_i) : (\mathcal{K}_{i-1}, T_{i-1}) \rightarrow (\mathcal{K}_i, T_i)$$

be T-additive functors. The composition

$$(F, \tau) = (F_2, \tau_2) \circ (F_1, \tau_1)$$

is the T-additive functor $(\mathcal{K}_0, T_0) \rightarrow (\mathcal{K}_2, T_2)$ defined as follows: the functor is $F := F_2 \circ F_1$, and the translation isomorphism

$$\tau : F \circ T_0 \xrightarrow{\sim} T_2 \circ F$$

is $\tau := \tau_2 \circ F_2(\tau_1)$.

Definition 5.1.5. Suppose $(\mathcal{K}, T_{\mathcal{K}})$ and $(\mathcal{L}, T_{\mathcal{L}})$ are T-additive categories, and

$$(F, \tau), (G, \nu) : (\mathcal{K}, T_{\mathcal{K}}) \rightarrow (\mathcal{L}, T_{\mathcal{L}})$$

are T-additive functors. A *morphism of T-additive functors*

$$\eta : (F, \tau) \rightarrow (G, \nu)$$

is a morphism of functors $\eta : F \rightarrow G$, such that for every object $M \in \mathbf{K}$ this diagram in \mathbf{L} is commutative:

$$\begin{array}{ccc} F(\mathbf{T}_{\mathbf{K}}(M)) & \xrightarrow{\tau_M} & \mathbf{T}_{\mathbf{L}}(F(M)) \\ \eta_{\mathbf{T}_{\mathbf{K}}(M)} \downarrow & & \downarrow \mathbf{T}_{\mathbf{L}}(\eta_M) \\ G(\mathbf{T}_{\mathbf{K}}(M)) & \xrightarrow{\nu_M} & \mathbf{T}_{\mathbf{L}}(G(M)) \end{array} .$$

Definition 5.1.6. Let (\mathbf{K}, \mathbf{T}) be an additive category with translation. A *triangle* in (\mathbf{K}, \mathbf{T}) is a diagram

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathbf{T}(L)$$

in \mathbf{K} .

Definition 5.1.7. Let (\mathbf{K}, \mathbf{T}) be a \mathbf{T} -additive category. Suppose

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathbf{T}(L)$$

and

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathbf{T}(L')$$

are triangles in (\mathbf{K}, \mathbf{T}) . A *morphism of triangles* between them is a commutative diagram

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & \mathbf{T}(L) \\ \phi \downarrow & & \downarrow \psi & & \downarrow \chi & & \downarrow \mathbf{T}(\phi) \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & \mathbf{T}(L') \end{array}$$

in \mathbf{K} .

The morphism of triangles (ϕ, ψ, χ) is called an isomorphism if ϕ, ψ and χ are all isomorphisms.

Remark 5.1.8. Why “triangle”? This is because sometimes a triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathbf{T}(L)$$

is written as a diagram

$$\begin{array}{ccc} & N & \\ \gamma \swarrow & & \nwarrow \beta \\ L & \xrightarrow{\alpha} & M \end{array}$$

But here γ is a morphism “of degree 1”.

5.2. Pretriangulated Categories.

Definition 5.2.1. A *pretriangulated category* is a \mathbf{T} -additive category (\mathbf{K}, \mathbf{T}) , equipped with a set of triangles called *distinguished triangles*. The following axioms have to be satisfied:

- (TR1) (a) Any triangle that is isomorphic to a distinguished triangle is also a distinguished triangle.
- (b) For every morphism $\alpha : L \rightarrow M$ in \mathbf{K} there is a distinguished triangle

$$L \xrightarrow{\alpha} M \rightarrow N \rightarrow \mathbf{T}(L).$$

(c) For every object M the triangle

$$M \xrightarrow{1_M} M \rightarrow 0 \rightarrow T(M)$$

is distinguished.

(TR2) A triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

is distinguished iff the triangle

$$M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L) \xrightarrow{-T(\alpha)} T(M)$$

is distinguished.

(TR3) Suppose

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\ \phi \downarrow & & \psi \downarrow & & & & \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L') \end{array}$$

is a commutative diagram in \mathbf{K} in which the rows are distinguished triangles. Then there exists a morphism $\chi : N \rightarrow N'$ such that the diagram

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\ \phi \downarrow & & \psi \downarrow & & \chi \downarrow & & T(\phi) \downarrow \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L') \end{array} .$$

is a morphism of triangles.

Remark 5.2.2. The numbering of the axioms we use is taken from [RD]; the numbering in [Sc], [KaSc1] [KaSc2] and [Ne] is different.

In the situation that we care about, namely $\mathbf{K} = \mathbf{K}(A, \mathbf{M})$, the distinguished triangles will be those triangles that are isomorphic, in $\mathbf{K}(A, \mathbf{M})$, to the standard triangles in $\mathbf{C}(A, \mathbf{M})$ from Definition 3.9.5. See Definition 5.4.3 below for the precise statement.

The object N in item (b) of axiom (TR1) is referred to as a *cone* on $\alpha : L \rightarrow M$. We should think of the cone as something combining “the cokernel” and “the kernel” of α .

Axiom (TR2) says that if we “turn” a distinguished triangle we remain with a distinguished triangle.

Axiom (TR3) says that a commutative square (ϕ, ψ) induces a morphism χ on the cones of the horizontal morphisms, that fits into a morphism of distinguished triangles (ϕ, ψ, χ) . Note however that the new morphism χ is *not unique*; in other words, *cones are not functorial*. This fact has some deep consequences in many applications. However, in the situations that will interest us, namely when $\mathbf{K} = \mathbf{K}(A, \mathbf{M})$, the cones come from the standard cones in $\mathbf{C}(A, \mathbf{M})$; and the standard cones in $\mathbf{C}(A, \mathbf{M})$ are functorial (Definition 3.9.6).

Remark 5.2.3. There is a fourth axiom in the literature, called the *octahedral axiom*; it is axiom (TR4) in [RD]. Keeping with the traditional usage, the name *triangulated category* is reserved for a pretriangulated category that also satisfies this extra axiom.

Because the octahedral axiom is extremely cumbersome to state (and prove), and also it does not play any role in the study of derived categories, we have decided to ignore it completely in our book.

For the role of the octahedral axiom in the structure of abstract triangulated categories, see the book [Ne]. It is not known whether the octahedral axiom is a consequence of the other axioms; there was a recent paper by Maccioca (arxiv:1506.00887) claiming that, but it had a fatal error in it.

The reader should not confuse the meaning of the name “pretriangulated category”, as used here, with the “pretriangulated DG category” from [BoKa]. See Remark 5.4.15.

For a category \mathcal{K} there is a canonical contravariant functor $\text{op} : \mathcal{K} \rightarrow \mathcal{K}^{\text{op}}$, that is the identity on objects, and reverses the arrows. Note that op is an anti-isomorphism of categories (i.e. a contravariant isomorphism), so its inverse op^{-1} is unique.

Proposition 5.2.4. *Let \mathcal{K} be a pretriangulated category, and let $\text{op} : \mathcal{K} \rightarrow \mathcal{K}^{\text{op}}$ be the canonical contravariant functor from \mathcal{K} to its opposite category. Define a translation T^{op} on \mathcal{K}^{op} by the formula $T^{\text{op}} := \text{op} \circ T^{-1} \circ \text{op}^{-1}$. The distinguished triangles in \mathcal{K}^{op} are defined to be the triangles*

$$N \xrightarrow{\text{op}(\beta)} M \xrightarrow{\text{op}(\alpha)} L \xrightarrow{\text{op}(-T^{-1}(\gamma))} T^{\text{op}}(N),$$

where $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ is any distinguished triangle in \mathcal{K} . Then $(\mathcal{K}^{\text{op}}, T^{\text{op}})$ is a pretriangulated category.

Proof. This is an exercise. (Hint: use the proof of Proposition 5.3.3 below.) \square

Proposition 5.2.5. *Let \mathcal{K} be a pretriangulated category. If*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

is a distinguished triangle in \mathcal{K} , then $\beta \circ \alpha = 0$.

Proof. By axioms (TR1) and (TR3) we have a commutative diagram

$$\begin{array}{ccccccc} L & \xrightarrow{1_L} & L & \longrightarrow & 0 & \longrightarrow & T(L) \\ 1_L \downarrow & & \downarrow \alpha & & \downarrow & & \downarrow T(1_L) \\ L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \end{array}.$$

We see that $\beta \circ \alpha$ factors through 0. \square

5.3. Triangulated and Cohomological Functors. Suppose \mathcal{K} and \mathcal{L} are T -additive categories, with translation functors $T_{\mathcal{K}}$ and $T_{\mathcal{L}}$ respectively. The notion of T -additive functor $F : \mathcal{K} \rightarrow \mathcal{L}$ was defined in Definition 5.1.3. In that definition we also introduced the notion of morphism $\eta : F \rightarrow G$ between T -additive functors.

Definition 5.3.1. Let \mathcal{K} and \mathcal{L} be pretriangulated categories.

- (1) A *triangulated functor* from \mathcal{K} to \mathcal{L} is a T -additive functor

$$(F, \tau) : \mathcal{K} \rightarrow \mathcal{L}$$

that satisfies this condition: for any distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T_{\mathcal{K}}(L)$$

in \mathbf{K} , the triangle

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N) \xrightarrow{\tau_L \circ F(\gamma)} T_L(F(L))$$

is a distinguished triangle in \mathbf{L} .

- (2) Suppose $(G, \nu) : \mathbf{K} \rightarrow \mathbf{L}$ is another triangulated functor. A *morphism of triangulated functors* $\eta : (F, \tau) \rightarrow (G, \nu)$ is a morphism of \mathbf{T} -additive functors, as in Definition 5.1.5 .

Sometimes we keep the translation isomorphism τ implicit, and refer to F as a triangulated functor.

Definition 5.3.2. Let \mathbf{K} be a pretriangulated category, and let \mathbf{M} be an abelian category. A *cohomological functor* $F : \mathbf{K} \rightarrow \mathbf{M}$ is an additive functor, such that for every distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

in \mathbf{K} , the sequence

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N)$$

is exact in \mathbf{M} .

Proposition 5.3.3. Let $F : \mathbf{K} \rightarrow \mathbf{M}$ be a cohomological functor, and let

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

be a distinguished triangle in \mathbf{K} . Then the sequence

$$\begin{aligned} \cdots \rightarrow F(T^i(L)) &\xrightarrow{F(T^i(\alpha))} F(T^i(M)) \xrightarrow{F(T^i(\beta))} F(T^i(N)) \xrightarrow{F(T^i(\gamma))} F(T^{i+1}(L)) \\ &\xrightarrow{F(T^{i+1}(\alpha))} F(T^{i+1}(M)) \rightarrow \cdots \end{aligned}$$

in \mathbf{M} is exact.

Proof. By axiom (TR2) we have distinguished triangles

$$T^i(L) \xrightarrow{(-1)^i \cdot T^i(\alpha)} T^i(M) \xrightarrow{(-1)^i \cdot T^i(\beta)} T^i(N) \xrightarrow{(-1)^i \cdot T^i(\gamma)} T^{i+1}(L),$$

$$T^i(M) \xrightarrow{(-1)^i \cdot T^i(\beta)} T^i(N) \xrightarrow{(-1)^i \cdot T^i(\gamma)} T^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot T^{i+1}(\alpha)} T^{i+1}(M)$$

and

$$T^i(N) \xrightarrow{(-1)^i \cdot T^i(\gamma)} T^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot T^{i+1}(\alpha)} T^{i+1}(M) \xrightarrow{(-1)^{i+1} \cdot T^{i+1}(\beta)} T^{i+1}(N).$$

Now use the definition, noting that multiplying morphisms in an exact sequence by -1 preserves exactness. \square

Proposition 5.3.4. Let \mathbf{K} be a pretriangulated category. For any $P \in \mathbf{K}$ the functors

$$\mathrm{Hom}_{\mathbf{K}}(-, P) : \mathbf{K}^{\mathrm{op}} \rightarrow \mathbf{Ab}$$

and

$$\mathrm{Hom}_{\mathbf{K}}(P, -) : \mathbf{K} \rightarrow \mathbf{Ab}$$

are cohomological functors.

Proof. We will prove the covariant statement; the contravariant statement is an immediate consequence, since

$$\mathrm{Hom}_{\mathbf{K}}(M, P) = \mathrm{Hom}_{\mathbf{K}^{\mathrm{op}}}(P, M),$$

and \mathbf{K}^{op} is pretriangulated (with the correct pretriangulated structure to make this true).

Consider a distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

in \mathbf{K} . We have to prove that the sequence

$$\mathrm{Hom}_{\mathbf{K}}(P, L) \xrightarrow{\mathrm{Hom}(1_P, \alpha)} \mathrm{Hom}_{\mathbf{K}}(P, M) \xrightarrow{\mathrm{Hom}(1_P, \beta)} \mathrm{Hom}_{\mathbf{K}}(P, N)$$

is exact. In view of Proposition 5.2.5, all we need to show is that for any $\psi : P \rightarrow M$ s.t. $\beta \circ \psi = 0$, there is some $\phi : P \rightarrow L$ s.t. $\psi = \alpha \circ \phi$. In a picture, we must show that the diagram below (solid arrows)

$$\begin{array}{ccccccc} P & \xrightarrow{1} & P & \longrightarrow & 0 & \longrightarrow & T(P) \\ \downarrow \phi & & \downarrow \psi & & \downarrow & & \downarrow T(\phi) \\ L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \end{array}$$

can be completed (dashed arrow). This is true by (TR2) (= turning) and (TR3) (= extending). \square

Proposition 5.3.5. *Let \mathbf{K} be a pretriangulated category, and let*

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\ \downarrow \phi & & \downarrow \psi & & \downarrow \chi & & \downarrow T(\phi) \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L') \end{array}$$

be a morphism of distinguished triangles. If ϕ and ψ are isomorphisms, then χ is also an isomorphism.

Proof. Take an arbitrary $P \in \mathbf{K}$, and let $F := \mathrm{Hom}_{\mathbf{K}}(P, -)$. We get a commutative diagram

$$\begin{array}{ccccccccc} F(L) & \xrightarrow{F(\alpha)} & F(M) & \xrightarrow{F(\beta)} & F(N) & \xrightarrow{F(\gamma)} & F(T(L)) & \xrightarrow{F(T(\alpha))} & F(T(M)) \\ \downarrow F(\phi) & & \downarrow F(\psi) & & \downarrow F(\chi) & & \downarrow F(T(\phi)) & & \downarrow F(T(\psi)) \\ F(L') & \xrightarrow{F(\alpha')} & F(M') & \xrightarrow{F(\beta')} & F(N') & \xrightarrow{F(\gamma')} & F(T(L')) & \xrightarrow{F(T(\alpha'))} & F(T(M')) \end{array}$$

in \mathbf{Ab} . By Proposition 5.3.4(2) the rows in the diagram are exact sequences. Since the other vertical arrows are isomorphisms, it follows that

$$F(\chi) : \mathrm{Hom}_{\mathbf{K}}(P, N) \rightarrow \mathrm{Hom}_{\mathbf{K}}(P, N')$$

is an isomorphism of abelian groups. By forgetting structure, we see that $F(\chi)$ is an isomorphism of sets.

We now use the Yoneda Lemma. Let us write $Y_N := \text{Hom}_K(-, N)$ and $Y_{N'} := \text{Hom}_K(-, N')$, viewed as functors $K^{\text{op}} \rightarrow \text{Set}$. For any object $P \in K$ we have isomorphisms of sets $Y_N(P) \cong F(N)$ and $Y_{N'}(P) \cong F(N')$. The calculation above shows that the morphism of functors $Y(\chi) : Y_N \rightarrow Y_{N'}$ is an isomorphism. According to Proposition 1.7.1(2), the morphism $\chi : N \rightarrow N'$ in K is an isomorphism. \square

Proposition 5.3.6. *Let K be a pretriangulated category, and let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

be a distinguished triangle in it. The two conditions below are equivalent:

- (i) $\alpha : L \rightarrow M$ is an isomorphism.
- (ii) $N \cong 0$.

Proof. Exercise. (Hint: use Proposition 5.3.5.) \square

Question 5.3.7. Let K and L be pretriangulated categories, and let $F : K \rightarrow L$ be an additive functor. Is it true that there is at most one isomorphism of functors $\tau : F \circ T_K \xrightarrow{\sim} T_L \circ F$ such that the pair (F, τ) is a triangulated functor?

5.4. The Homotopy Category is Pretriangulated. In this subsection we consider an abelian category M and a DG ring A (everything over the base ring \mathbb{K}). These ingredients give rise to the \mathbb{K} -linear DG category $\mathbf{C}(A, M)$ of DG A -module in M , as in Subsection 3.7.

The strict category $\mathbf{C}_{\text{str}}(A, M)$ and the homotopy category $\mathbf{K}(A, M)$ was introduced in Definition 3.7.5. Recall that these linear categories have the same objects as $\mathbf{C}(A, M)$. The morphisms groups are

$$\text{Hom}_{\mathbf{C}_{\text{str}}(A, M)}(M_0, M_1) = Z^0(\text{Hom}_{\mathbf{C}(A, M)}(M_0, M_1))$$

and

$$\text{Hom}_{\mathbf{K}(A, M)}(M_0, M_1) = H^0(\text{Hom}_{\mathbf{C}(A, M)}(M_0, M_1)).$$

Thus the morphisms $M_0 \rightarrow M_1$ in $\mathbf{K}(A, M)$ are the homotopy classes $\bar{\phi} : M_0 \rightarrow M_1$ of the morphisms $\phi : M_0 \rightarrow M_1$ in $\mathbf{C}_{\text{str}}(A, M)$.

Recall the full additive functor

$$(5.4.1) \quad P : \mathbf{C}_{\text{str}}(A, M) \rightarrow \mathbf{K}(A, M)$$

from Definition 3.4.4, that is the identity on objects, and on morphisms it is $P(\phi) := \bar{\phi}$.

Consider the translation functor T from Definition 3.8.8. Since T is a DG functor from $\mathbf{C}(A, M)$ to itself (see Corollary 3.8.9), it restricts to a linear functor from $\mathbf{C}_{\text{str}}(A, M)$ to itself, and it induces a linear functor \bar{T} from $\mathbf{K}(A, M)$ to itself, such that $P \circ T = \bar{T} \circ P$.

Proposition 5.4.2.

- (1) *The category $\mathbf{C}_{\text{str}}(A, M)$, equipped with the translation functor T , is a T -additive category.*
- (2) *The category $\mathbf{K}(A, M)$, equipped with the translation functor \bar{T} , is a T -additive category.*
- (3) *Let $\tau : P \circ T \xrightarrow{\sim} \bar{T} \circ P$ be equality. Then the pair*

$$(P, \tau) : \mathbf{C}_{\text{str}}(A, M) \rightarrow \mathbf{K}(A, M)$$

is a T -additive functor.

Proof. (1) We need to prove that $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ is additive. Of course the zero complex is a zero object. Next we consider finite direct sums. Let M_1, \dots, M_r be a finite collection of objects in $\mathbf{C}(A, \mathbf{M})$. Each M_i is a DG A -module in \mathbf{M} , and we write it as $M_i = \{M_i^j\}_{j \in \mathbb{Z}}$. In each degree j the direct sum $M^j := \bigoplus_{i=1}^r M_i^j$ exists in \mathbf{M} . Let $M := \{M^j\}_{j \in \mathbb{Z}}$ be the resulting graded object in \mathbf{M} . The differential $d_M : M^j \rightarrow M^{j+1}$ exists by the universal property of direct sums; so we obtain a complex $M \in \mathbf{C}(\mathbf{M})$. The DG A -module structure on M is defined similarly: for $a \in A^k$, there is an induced degree k morphism $f(a) : M \rightarrow M$ in $\mathbf{C}(\mathbf{M})$. Thus M becomes an object of $\mathbf{C}(A, \mathbf{M})$. But the embeddings $e_i : M_i \rightarrow M$ are strict morphisms, so $(M, \{e_i\})$ is a coproduct of the collection $\{M_i\}$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$.

(2) Now consider the category $\mathbf{K}(A, \mathbf{M})$. Because the functor $P : \mathbf{C}_{\text{str}}(A, \mathbf{M}) \rightarrow \mathbf{K}(A, \mathbf{M})$ is additive, and is bijective on objects, part (1) above and Proposition 2.4.2 say that $\mathbf{K}(A, \mathbf{M})$ is an additive category.

(3) Clear. \square

From now on we denote by T , instead of by \bar{T} , the translation functor of $\mathbf{K}(A, \mathbf{M})$.

Definition 5.4.3. A triangle

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} T(L)$$

in $\mathbf{K}(A, \mathbf{M})$ is said to be a *distinguished triangle* if there is a standard triangle

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} T(L')$$

in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, as in Definition 3.9.5, and an isomorphism of triangles

$$\begin{array}{ccccccc} L' & \xrightarrow{P(\alpha')} & M' & \xrightarrow{P(\beta')} & N' & \xrightarrow{P(\gamma')} & T(L') \\ \bar{\phi} \downarrow & & \bar{\psi} \downarrow & & \bar{\chi} \downarrow & & T(\bar{\phi}) \downarrow \\ L & \xrightarrow{\bar{\alpha}} & M & \xrightarrow{\bar{\beta}} & N & \xrightarrow{\bar{\gamma}} & T(L) \end{array} .$$

in $\mathbf{K}(A, \mathbf{M})$.

Theorem 5.4.4. *The T -additive category $\mathbf{K}(A, \mathbf{M})$, with the set of distinguished triangles defined above, is a pretriangulated category.*

The proof is after three lemmas.

Lemma 5.4.5. *Let $M \in \mathbf{C}(A, \mathbf{M})$, and consider the cone $N := \text{Cone}(1_M)$. Then the DG module N is null-homotopic, i.e. $0 \rightarrow N$ is an isomorphism in $\mathbf{K}(A, \mathbf{M})$.*

Proof. We shall exhibit a homotopy θ from 0_N to 1_N . Recall from Subsection 3.9 that

$$N = \text{Cone}(1_M) = M \oplus T(M) = \begin{bmatrix} M \\ T(M) \end{bmatrix}$$

as graded modules, with differential whose matrix presentation is

$$d_N = \begin{bmatrix} d_M & t_M^{-1} \\ 0 & d_{T(M)} \end{bmatrix} .$$

And by the definition in Subsection 3.8 we have

$$d_{T(M)} = -t_M \circ d_M \circ t_M^{-1} .$$

Define $\theta : N \rightarrow N$ to be the degree -1 morphism with matrix presentation

$$\theta := \begin{bmatrix} 0 & 0 \\ t_M & 0 \end{bmatrix}.$$

Then, using the formulas above for d_N and $d_{T(M)}$, we get

$$d_N \circ \theta + \theta \circ d_N = \begin{bmatrix} 1_M & 0 \\ 0 & 1_{T(M)} \end{bmatrix} = 1_N.$$

□

Exercise 5.4.6. Here is a generalization of Lemma 5.4.5. Consider a morphism $\phi : M_0 \rightarrow M_1$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. Show that the three conditions below are equivalent:

- (i) ϕ is a homotopy equivalence.
- (ii) $\bar{\phi}$ is an isomorphism in $\mathbf{K}(A, \mathbf{M})$.
- (iii) The DG module $\text{Cone}(\phi)$ is null-homotopic.

Try to do this directly, not using Proposition 5.3.4(2) and Theorem 5.4.4.

The next lemma is based on [KaSc1, Lemma 1.4.2].

Lemma 5.4.7. Consider a morphism $\alpha : L \rightarrow M$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, the standard triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

associated to α , and the standard triangle

$$M \xrightarrow{\beta} N \xrightarrow{\phi} P \xrightarrow{\psi} T(M)$$

associated to β , all in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. So $N = \text{Cone}(\alpha)$ and $P = \text{Cone}(\beta)$. There is a morphism $\rho : T(L) \rightarrow P$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ s.t. $\bar{\rho}$ is an isomorphism in $\mathbf{K}(A, \mathbf{M})$, and the diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\bar{\beta}} & N & \xrightarrow{\bar{\gamma}} & T(L) & \xrightarrow{-T(\bar{\alpha})} & T(M) \\ \bar{1}_M \downarrow & & \bar{1}_N \downarrow & & \bar{\rho} \downarrow & & \bar{1}_{T(M)} \downarrow \\ M & \xrightarrow{\bar{\beta}} & N & \xrightarrow{\bar{\phi}} & P & \xrightarrow{\bar{\psi}} & T(M) \end{array}$$

commutes in $\mathbf{K}(A, \mathbf{M})$.

Proof. Note that $N = M \oplus T(L)$ and $P = N \oplus T(M) = M \oplus T(L) \oplus T(M)$ as graded module. Thus P and d_P have the following matrix presentations:

$$P = \begin{bmatrix} M \\ T(L) \\ T(M) \end{bmatrix}, \quad d_P = \begin{bmatrix} d_M & \alpha \circ t_L^{-1} & t_M^{-1} \\ 0 & d_{T(L)} & 0 \\ 0 & 0 & d_{T(M)} \end{bmatrix}.$$

Define morphisms $\rho : T(L) \rightarrow P$ and $\chi : P \rightarrow T(L)$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ by the matrix presentations

$$\rho := \begin{bmatrix} 0 \\ 1_{T(L)} \\ -T(\alpha) \end{bmatrix}, \quad \chi := \begin{bmatrix} 0 & 1_{T(L)} & 0 \end{bmatrix}.$$

Direct calculations show that:

- $\chi \circ \rho = 1_{T(L)}$.
- $\rho \circ \gamma = \rho \circ \chi \circ \phi$.
- $\psi \circ \rho = -T(\alpha)$.

It remains to prove that $\rho \circ \chi$ is homotopic to 1_P . Define a degree -1 morphism $\theta : P \rightarrow P$ by the matrix

$$\theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_M & 0 & 0 \end{bmatrix}.$$

Then a direct calculation, using the equalities

$$t_M \circ d_M + d_{T(M)} \circ t_M = 0$$

and

$$T(\alpha) = t_M \circ \alpha \circ t_L^{-1}$$

gives

$$\theta \circ d_P + d_P \circ \theta = 1_P - \rho \circ \chi.$$

□

Lemma 5.4.8. *Consider a standard triangle*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. For any integer k , the triangle

$$T^k(L) \xrightarrow{T^k(\alpha)} T^k(M) \xrightarrow{T^k(\beta)} T^k(N) \xrightarrow{(-1)^k \cdot T^k(\gamma)} T^{k+1}(L)$$

is isomorphic, in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, to a standard triangle.

Proof. Combine Corollary 3.8.9, Corollary 4.3.8 with $F = T$, and Proposition 4.2.4. □

Proof of Theorem 5.4.4. (TR1): By definition the set of distinguished triangles in $\mathbf{K}(A, \mathbf{M})$ is closed under isomorphisms. This establishes item (a).

As for item (b): consider any morphism $\bar{\alpha} : L \rightarrow M$ in $\mathbf{K}(A, \mathbf{M})$. It is represented by a morphism $\alpha : L \rightarrow M$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. Take the standard triangle on α in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. Its image in $\mathbf{K}(A, \mathbf{M})$ has the desired property.

Finally, Lemma 5.4.5 shows that the triangle

$$M \xrightarrow{\bar{1}_M} M \rightarrow 0 \rightarrow T(M)$$

is isomorphic in $\mathbf{K}(A, \mathbf{M})$ to the triangle

$$M \xrightarrow{\bar{1}_M} M \xrightarrow{\bar{e}} \text{Cone}(1_M) \xrightarrow{\bar{p}} T(M).$$

The latter is the image of a standard triangle, and so it is distinguished.

(TR2): Consider the triangles

$$(5.4.9) \quad L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} T(L)$$

and

$$(5.4.10) \quad M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} T(L) \xrightarrow{-T(\bar{\alpha})} T(M)$$

in $\mathbf{K}(A, \mathbf{M})$. If (5.4.9) is distinguished, then by Lemma 5.4.7 so is (5.4.10).

Conversely, if (5.4.10) is distinguished, then by turning it 5 times, and using the previous step (namely by Lemma 5.4.7), we see that the triangle

$$T^2(L) \xrightarrow{T^2(\bar{\alpha})} T^2(M) \xrightarrow{T^2(\bar{\beta})} T^2(N) \xrightarrow{T^2(\bar{\gamma})} T^3(L)$$

is distinguished. According to Lemma 5.4.8 (with $k = -2$), the triangle gotten by applying T^{-2} to this is distinguished. But this is just the triangle (5.4.9).

(TR3): Consider a commutative diagram in $\mathbf{K}(A, \mathbf{M})$:

$$(5.4.11) \quad \begin{array}{ccccccc} \bar{L} & \xrightarrow{\bar{\alpha}} & \bar{M} & \xrightarrow{\bar{\beta}} & \bar{N} & \xrightarrow{\bar{\gamma}} & T(\bar{L}) \\ \bar{\phi} \downarrow & & \bar{\psi} \downarrow & & & & \\ \bar{L}' & \xrightarrow{\bar{\alpha}'} & \bar{M}' & \xrightarrow{\bar{\beta}'} & \bar{N}' & \xrightarrow{\bar{\gamma}'} & T(\bar{L}') \end{array}$$

where the horizontal triangles are distinguished. By definition the rows in (5.4.11) are isomorphic in $\mathbf{K}(A, \mathbf{M})$ to the images under the functor P of standard triangles in $\mathbf{C}(A, \mathbf{M})$. These are the rows in diagram (5.4.12) below. The vertical morphisms in (5.4.11) are also induced from morphisms in $\mathbf{C}(A, \mathbf{M})$, i.e. $\bar{\phi} = P(\phi)$ and $\bar{\psi} = P(\psi)$. Thus (5.4.11) is isomorphic to the image under P of the following diagram:

$$(5.4.12) \quad \begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\ \phi \downarrow & & \psi \downarrow & & & & \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L') \end{array}$$

Warning: the diagram (5.4.12) is only commutative up to homotopy in $\mathbf{C}(A, \mathbf{M})$.

Since the rows in (5.4.12) are standard triangles (see Definition 3.9.5), the objects N and N' are cones: $N = \text{Cone}(\alpha)$ and $N' = \text{Cone}(\alpha')$. The commutativity up to homotopy of this diagram means that there is a degree -1 morphism $\theta : L \rightarrow M'$ in $\mathbf{C}(A, \mathbf{M})$ such that

$$\alpha' \circ \phi = \psi \circ \alpha + d(\theta).$$

Define the morphism

$$\chi : N = \begin{bmatrix} M \\ T(L) \end{bmatrix} \rightarrow N' = \begin{bmatrix} M' \\ T(L') \end{bmatrix}$$

by the matrix presentation

$$\chi := \begin{bmatrix} \psi & \theta \circ \iota_L^{-1} \\ 0 & T(\phi) \end{bmatrix}.$$

An easy calculation shows that χ is a morphism in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, and that there are equalities $T(\phi) \circ \gamma = \gamma' \circ \chi$ and $\chi \circ \beta = \beta' \circ \psi$. Therefore, when we apply the functor P , and conjugate by the original isomorphism between (5.4.11) and the image of (5.4.12), we obtain a commutative diagram

$$\begin{array}{ccccccc} \bar{L} & \xrightarrow{\bar{\alpha}} & \bar{M} & \xrightarrow{\bar{\beta}} & \bar{N} & \xrightarrow{\bar{\gamma}} & T(\bar{L}) \\ \bar{\phi} \downarrow & & \bar{\psi} \downarrow & & \bar{\chi} \downarrow & & T(\bar{\phi}) \downarrow \\ \bar{L}' & \xrightarrow{\bar{\alpha}'} & \bar{M}' & \xrightarrow{\bar{\beta}'} & \bar{N}' & \xrightarrow{\bar{\gamma}'} & T(\bar{L}') \end{array}$$

in $\mathbf{K}(A, \mathbf{M})$, where $\bar{\chi}$ is conjugate to $P(\chi)$. \square

We now add a second DG ring B , and a second additive category \mathbf{N} . DG functors were introduced in Subsection 3.5.

Consider a DG functor

$$F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N}).$$

From Theorem 4.2.3 we know that the translation isomorphism is an isomorphism of DG functors

$$\tau_F : F \circ T_{A, \mathbf{M}} \xrightarrow{\cong} T_{B, \mathbf{N}} \circ F.$$

Therefore, when we pass to the homotopy categories, and writing $\bar{F} := \text{Ho}(F)$, we get a T -additive functor

$$(\bar{F}, \bar{\tau}_F) : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{K}(B, \mathbf{N}).$$

Theorem 5.4.13. *Let*

$$F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N})$$

be a DG functor, with translation isomorphism τ_F . Then the T -additive functor

$$(\bar{F}, \bar{\tau}_F) : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{K}(B, \mathbf{N})$$

is a triangulated functor.

Proof. Take a distinguished triangle

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} T(L)$$

in $\mathbf{K}(A, \mathbf{M})$. Since we are only interested in triangles up to isomorphism, we can assume that this is the image under the functor P of a standard triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. According to Theorem 4.3.7 and Corollary 4.3.8, there is a standard triangle

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} T(L')$$

in $\mathbf{C}_{\text{str}}(B, \mathbf{N})$, and a commutative diagram

$$\begin{array}{ccccccc} F(L) & \xrightarrow{F(\alpha)} & F(M) & \xrightarrow{F(\beta)} & F(N) & \xrightarrow{\tau_{F,L} \circ F(\gamma)} & T(F(L)) \\ \downarrow \phi & & \downarrow \psi & & \downarrow \chi & & \downarrow T(\phi) \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L') \end{array}$$

in $\mathbf{C}_{\text{str}}(B, \mathbf{N})$, in which the vertical arrows are isomorphisms. (Actually, we can take $L' = F(L)$, $\phi = \text{id}_{F(L)}$, etc.) After applying the functor P to this diagram, we see that the condition in Definition 5.3.1(1) is satisfied. \square

Corollary 5.4.14. *For any integer k , the pair $(T^k, (-1)^k \cdot \text{id}_{T^{k+1}})$ is a triangulated functor from $\mathbf{K}(A, \mathbf{M})$ to itself.*

Proof. Combine Theorems 5.4.13 and Proposition 4.2.4. \square

Remark 5.4.15. In [BoKa], Bondal and Kapranov introduce the concept of *pretriangulated DG category*. This is a DG category \mathcal{C} for which the homotopy category $\mathrm{Ho}(\mathcal{C})$ is canonically triangulated (the details of the definition are too complicated to mention here). Our DG categories $\mathbf{C}(A, \mathbf{M})$ are pretriangulated in the sense of [BoKa]; but they have a lot more structure (e.g. the objects have cohomologies too).

Suppose \mathcal{C} and \mathcal{C}' are pretriangulated DG categories. In [BoKa] there is a (rather complicated) definition of *pre-exact DG functor* $F : \mathcal{C} \rightarrow \mathcal{C}'$. It is stated there that if F is a pre-exact DG functor, then $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C}')$ is a triangulated functor. This is analogous to our Theorem 5.4.13. Presumably, Theorems 4.2.3 and 4.3.7 imply that any DG functor $F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(A', \mathbf{M}')$ is pre-exact in the sense of [BoKa]; but we did not verify this.

6. LOCALIZATION OF CATEGORIES

Most of this section is devoted to the general theory of *Ore localization of categories*. In the last subsection we talk about localization of a pretriangulated category \mathbf{K} with respect to a denominator set of cohomological origin $\mathbf{S} \subseteq \mathbf{K}$.

6.1. The Formalism of Localization. We will start with a category \mathbf{A} , without even assuming it is linear. Still we use the notation \mathbf{A} , because it will be suggestive to think about a linear category \mathbf{A} with a single object, which is just a ring A . The reason is that our localization procedure is the same as that in noncommutative ring theory (the only change being that we allow multiple objects).

The emphasis will be on morphisms rather than on objects. Thus it will be convenient to write

$$\mathbf{A}(M, N) := \text{Hom}_{\mathbf{A}}(M, N)$$

for $M, N \in \text{Ob}(\mathbf{A})$. We sometimes use the notation $a \in \mathbf{A}$ for a morphism $a \in \mathbf{A}(M, N)$, leaving the objects implicit. When we write $b \circ a$ for $a, b \in \mathbf{A}$, we implicitly mean that these morphisms are composable.

For heuristic purposes, we can think of \mathbf{A} as a linear category (e.g. living inside some category of modules), with objects M, N, \dots . For any given object M , we then have a genuine ring $\mathbf{A}(M) := \mathbf{A}(M, M)$.

Definition 6.1.1. Let \mathbf{A} be a category. A *multiplicatively closed set of morphisms* in \mathbf{A} is a subcategory $\mathbf{S} \subseteq \mathbf{A}$ such that $\text{Ob}(\mathbf{S}) = \text{Ob}(\mathbf{A})$.

In other words, for any pair of objects $M, N \in \mathbf{A}$ there is a subset $\mathbf{S}(M, N) \subseteq \mathbf{A}(M, N)$, such that $1_M \in \mathbf{S}(M, M)$, and such that for any $s \in \mathbf{S}(L, M)$ and $t \in \mathbf{S}(M, N)$, the composition $t \circ s \in \mathbf{S}(L, N)$.

Using our shorthand, we can write the definition like this: $1_M \in \mathbf{S}$, and $s, t \in \mathbf{S}$ implies $t \circ s \in \mathbf{S}$.

If $\mathbf{A} = A$ is a single object linear category, namely a ring, then $\mathbf{S} = S$ is a multiplicatively closed set in the sense of ring theory.

There are various notions of localization in the literature. We restrict attention to two of them.

Definition 6.1.2. Let \mathbf{S} be a multiplicatively closed set of morphisms in a category \mathbf{A} . A *localization* of \mathbf{A} with respect to \mathbf{S} is a pair $(\mathbf{A}_{\mathbf{S}}, Q)$, consisting of a category $\mathbf{A}_{\mathbf{S}}$ and a functor $Q : \mathbf{A} \rightarrow \mathbf{A}_{\mathbf{S}}$, called the localization functor, having the following properties:

- (L1) There is equality $\text{Ob}(\mathbf{A}_{\mathbf{S}}) = \text{Ob}(\mathbf{A})$, and Q is the identity on objects.
- (L2) For every $s \in \mathbf{S}$, the morphism $Q(s) \in \mathbf{A}_{\mathbf{S}}$ is invertible (i.e. it is an isomorphism).
- (L3) Suppose \mathbf{B} is a category, and $F : \mathbf{A} \rightarrow \mathbf{B}$ is a functor such that $F(s)$ is invertible for every $s \in \mathbf{S}$. Then there is a unique functor $F_{\mathbf{S}} : \mathbf{A}_{\mathbf{S}} \rightarrow \mathbf{B}$ such that $F_{\mathbf{S}} \circ Q = F$ as functors $\mathbf{A} \rightarrow \mathbf{B}$.

In a commutative diagram:

$$\begin{array}{ccccc} \mathbf{S} & \xrightarrow{\text{inc}} & \mathbf{A} & \xrightarrow{F} & \mathbf{B} \\ & & \downarrow Q & \nearrow F_{\mathbf{S}} & \\ & & \mathbf{A}_{\mathbf{S}} & & \end{array}$$

In the ring case, $F : A \rightarrow B$ is a ring homomorphism, etc.

Proposition 6.1.3. *A localization (in the sense of Definition 6.1.2) is unique up to a unique isomorphism. Namely if (A'_S, Q') is another localization, then there is a unique functor $G : A_S \rightarrow A'_S$ which is the identity on objects, bijective on morphisms, and $G \circ Q = Q'$.*

Proof. Exercise. □

A localization in this general sense always exists, but often it is of little value, because there is no practical way to describe the morphisms in it.

6.2. Ore Localization. There is a better notion of localization. The references here are [RD], [GaZi], [We], [KaSc1], [Ste] and [Row]. The first four references talk about localization of categories; and the last two talk about noncommutative rings. It seems that historically, this *noncommutative calculus of fractions* was discovered by Ore and Asano in ring theory, around 1930. There was progress in the categorical side, notably by Gabriel around 1960.

In this subsection we mostly follow the treatment of [Ste]; but we sometimes use diagrams instead of formulas involving letters – this is the only way the author was able to understand the proofs!

Definition 6.2.1. Let S be a multiplicatively closed set of morphisms in a category A . A *right Ore localization* of A with respect to S is a pair (A_S, Q) , consisting of a category A_S and a functor $Q : A \rightarrow A_S$, having the following properties:

- (RO1) There is equality $\text{Ob}(A_S) = \text{Ob}(A)$, and Q is the identity on objects.
- (RO2) For every $s \in S$, the morphism $Q(s) \in A_S$ is an isomorphism.
- (RO3) Every morphism $q \in A_S$ can be written as $q = Q(a) \circ Q(s)^{-1}$ for some $a \in A$ and $s \in S$.
- (RO4) Suppose $a, b \in A$ satisfy $Q(a) = Q(b)$. Then $a \circ s = b \circ s$ for some $s \in S$.

The letters “RO” stand for “right Ore”. We refer to the expression $q = Q(a) \circ Q(s)^{-1}$ as a *right fraction representation* of q .

Remark 6.2.2. There is an obvious notion of *left Ore localization*, with properties (LO1)-(LO4) that are identical to (RO1)-(RO4) respectively, except that in the last two the compositions are reversed: $q = Q(s)^{-1} \circ Q(a)$ and $s \circ a = s \circ b$. The results to follow in this section all have “left” versions, with identical proofs (just a matter of reversing some arrows or compositions), and so they will be omitted.

To reinforce the last remark, we give:

Proposition 6.2.3. *Let S be a multiplicatively closed set in a category A , and let $Q : A \rightarrow A_S$ be a functor. Prove that $Q : A \rightarrow A_S$ is a right Ore localization of A with respect to S if and only if $Q^{\text{op}} : A^{\text{op}} \rightarrow (A^{\text{op}})_{S^{\text{op}}}$ is a left Ore localization of A^{op} with respect to S^{op} .*

Exercise 6.2.4. Prove Proposition 6.2.3.

Lemma 6.2.5. *Let (A_S, Q) be a right Ore localization, let $a_1, a_2 \in A$ and $s_1, s_2 \in S$. The following conditions are equivalent:*

- (i) $Q(a_1) \circ Q(s_1)^{-1} = Q(a_2) \circ Q(s_2)^{-1}$ in A_S .
- (ii) There are $b_1, b_2 \in A$ s.t. $a_1 \circ b_1 = a_2 \circ b_2$, and $s_1 \circ b_1 = s_2 \circ b_2 \in S$.

Proof. (ii) \Rightarrow (i): Since $Q(s_i)$ and $Q(s_i \circ b_i)$ are invertible, it follows that $Q(b_i)$ are invertible. So

$$\begin{aligned} Q(a_1) \circ Q(s_1)^{-1} &= Q(a_1) \circ Q(b_1) \circ Q(b_1)^{-1} \circ Q(s_1)^{-1} \\ &= Q(a_2) \circ Q(b_2) \circ Q(b_2)^{-1} \circ Q(s_2)^{-1} = Q(a_2) \circ Q(s_2)^{-1}. \end{aligned}$$

(i) \Rightarrow (ii): By property (RO3) there are $c \in A$ and $u \in S$ s.t.

$$(6.2.6) \quad Q(s_2)^{-1} \circ Q(s_1) = Q(c) \circ Q(u)^{-1}.$$

Rewriting this equation we get

$$(6.2.7) \quad Q(s_1 \circ u) = Q(s_2 \circ c).$$

It is given that

$$Q(a_1) = Q(a_2) \circ Q(s_2)^{-1} \circ Q(s_1).$$

Plugging (6.2.6) into it we obtain

$$Q(a_1) = Q(a_2) \circ Q(c) \circ Q(u)^{-1}.$$

Rearranging this equation we get

$$(6.2.8) \quad Q(a_1 \circ u) = Q(a_2 \circ c).$$

By property (RO4) there is $v \in S$ s.t.

$$a_1 \circ u \circ v = a_2 \circ c \circ v.$$

Likewise, from equation (6.2.7) and property (RO4), there is $v' \in S$ s.t.

$$s_1 \circ u \circ v' = s_2 \circ c \circ v'.$$

Again using property (RO3), there are $d \in A$ and $w \in S$ s.t.

$$Q(v)^{-1} \circ Q(v') = Q(d) \circ Q(w)^{-1}.$$

Rearranging we get

$$Q(v' \circ w) = Q(v \circ d).$$

By property (RO4) there is $w' \in S$ s.t.

$$v' \circ w \circ w' = v \circ d \circ w'.$$

Define

$$b_1 := u \circ v \circ d \circ w', \quad b_2 := c \circ v \circ d \circ w'.$$

Then

$$\begin{aligned} s_1 \circ b_1 &= s_1 \circ u \circ v \circ d \circ w' = s_1 \circ u \circ v' \circ w \circ w' \\ &= s_2 \circ c \circ v' \circ w \circ w' = s_2 \circ b_2, \end{aligned}$$

and it is in S . Also

$$a_1 \circ b_1 = a_1 \circ u \circ v \circ d \circ w' = a_2 \circ c \circ v \circ d \circ w' = a_2 \circ b_2.$$

□

Proposition 6.2.9. *A right Ore localization (A_S, Q) is a localization in the sense of Definition 6.1.2.*

Proof. Say \mathbf{B} is a category, and $F : \mathbf{A} \rightarrow \mathbf{B}$ is a functor such that $F(s)$ is an isomorphism for every $s \in \mathbf{S}$.

The uniqueness of a functor $F_S : \mathbf{A}_S \rightarrow \mathbf{B}$ satisfying $F_S \circ Q = F$ is clear from property (RO3). We have to prove existence.

Define F_S to be F on objects, and

$$F_S(q) := F(a_1) \circ F(s_1)^{-1},$$

where

$$q = Q(a_1) \circ Q(s_1)^{-1} \in \mathbf{A}_S, \quad a_1 \in \mathbf{A}, \quad s_1 \in \mathbf{S}$$

is any presentation of q as a right fraction, that exists by (RO3). We have to prove that this is well defined. So suppose that $q = Q(a_2) \circ Q(s_2)^{-1}$ is another presentation of q . Let $b_1, b_2 \in \mathbf{A}$ be as in Lemma 6.2.5. Since $F(s_i)$ and $F(s_i \circ b_i)$ are invertible, then so is $F(b_i)$. We get

$$F(a_2) = F(a_1) \circ F(b_1) \circ F(b_2)^{-1}$$

and

$$F(s_2) = F(s_1) \circ F(b_1) \circ F(b_2)^{-1}.$$

Hence

$$F(a_2) \circ F(s_2)^{-1} = F(a_1) \circ F(s_1)^{-1}.$$

It remains to prove that F_S is a functor. Since the identity 1_M of the object M in \mathbf{A}_S can be presented as $1_M = Q(1_M) \circ Q(1_M)^{-1}$, we see that $F_S(1_M) = 1_{F(M)}$.

Next let q_1 and q_2 be morphisms in \mathbf{A}_S , such that composition $q_2 \circ q_1$ exists (i.e. the target of q_1 is the source of q_2). We have to show that $F_S(q_2 \circ q_1)$ equals $F_S(q_2) \circ F_S(q_1)$. Choose presentations $q_i = Q(a_i) \circ Q(s_i)^{-1}$, so that

$$(6.2.10) \quad F_S(q_2) \circ F_S(q_1) = F(a_2) \circ F(s_2)^{-1} \circ F(a_1) \circ F(s_1)^{-1}.$$

By property (RO3) there is a right fraction presentation

$$(6.2.11) \quad Q(s_2)^{-1} \circ Q(a_1) = Q(b) \circ Q(t)^{-1}$$

for some $b \in \mathbf{A}$ and $t \in \mathbf{S}$. Because

$$Q(a_1 \circ t) = Q(s_2 \circ b),$$

by (RO4) there is some $r \in \mathbf{S}$ such that

$$a_1 \circ t \circ r = s_2 \circ b \circ r.$$

Therefore

$$F(a_1 \circ t \circ r) = F(s_2 \circ b \circ r),$$

which implies, by canceling the invertible morphism $F(r)$ and rearranging, that

$$(6.2.12) \quad F(s_2)^{-1} \circ F(a_1) = F(b) \circ F(t)^{-1}$$

in \mathbf{B} .

Let us continue. Using equation (6.2.11) we have

$$\begin{aligned} q_2 \circ q_1 &= Q(a_2) \circ Q(s_2)^{-1} \circ Q(a_1) \circ Q(s_1)^{-1} \\ &= Q(a_2) \circ Q(b) \circ Q(t)^{-1} \circ Q(s_1)^{-1} = Q(a_2 \circ b) \circ Q(s_1 \circ t)^{-1}. \end{aligned}$$

Using this presentation of $q_2 \circ q_1$, and the equality (6.2.12), we obtain

$$\begin{aligned} F_S(q_2 \circ q_1) &= F(a_2 \circ b) \circ F(s_1 \circ t)^{-1} = F(a_2) \circ F(b) \circ F(t)^{-1} \circ F(s_1)^{-1} \\ &= F(a_2) \circ F(s_2)^{-1} \circ F(a_1) \circ F(s_1)^{-1}. \end{aligned}$$

This is the same as (6.2.10). \square

Corollary 6.2.13. *Let S be a multiplicatively closed set of morphisms in a category A . Assume that (A_S, Q) and (A'_S, Q') are either right Ore localizations or left Ore localizations of A with respect to S . Then there is a unique isomorphism of localizations*

$$(A_S, Q) \cong (A'_S, Q'),$$

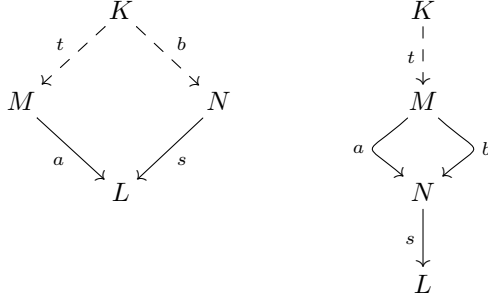
as in Proposition 6.1.3.

Proof. By Proposition 6.2.9 (in its right or left versions, as the case may be), both (A_S, Q) and (A'_S, Q') are localizations in the sense of Definition 6.1.2. Hence, by Proposition 6.1.3, there is a unique isomorphism $(A_S, Q) \cong (A'_S, Q')$. \square

Definition 6.2.14. Let S be a multiplicatively closed set of morphisms in a category A . We say that S is a *right denominator set* if it satisfies these two conditions:

- (RD1) (Right Ore condition) Given $a \in A$ and $s \in S$, there exist $b \in A$ and $t \in S$ such that $a \circ t = s \circ b$.
- (RD2) (Right Cancellation condition) Given $a, b \in A$ and $s \in S$ such that $s \circ a = s \circ b$, there exists $t \in S$ such that $a \circ t = b \circ t$.

In commutative diagrams:



There is a similar left version of this definition, with conditions (LD1) and (LD2). Here is the main theorem regarding Ore localization.

Theorem 6.2.15. *The following conditions are equivalent for a category A and a multiplicatively closed set of morphisms $S \subseteq A$.*

- (i) *The right Ore localization (A_S, Q) exists.*
- (ii) *S is a right denominator set.*

The proof of Theorem 6.2.15 is after some preparation. The hard part is proving that (ii) \Rightarrow (i). The general idea is the same as in commutative localization: we consider the set of pairs of morphisms $A \times S$, and define a relation \sim on it, with the hope that this is an equivalence relation, and that the quotient set A_S will be a category, and it will have the desired properties.

Let's assume that S is a right denominator set. For any $M, N \in \text{Ob}(A)$ consider the set

$$(A \times S)(M, N) := \coprod_{L \in \text{Ob}(A)} A(L, N) \times S(L, M).$$

Remark 6.2.16. The set $(A \times S)(M, N)$ could be big, namely not an element of the initial universe U . This would require the introduction of a larger universe, say V , in which U is an element. And the resulting category A_S will be a V -category.

We will ignore this issue. Moreover, in many cases of interest (derived categories where there are DG enhancements, such as the K-injective enhancement), there will be an alternative presentation of A_S as a U -category. We will refer to this when we get to it.

An element $(a, s) \in (A \times S)(M, N)$ can be pictured as a diagram

$$(6.2.17) \quad \begin{array}{ccc} & L & \\ s \swarrow & & \searrow a \\ M & & N \end{array}$$

in A . This diagram will eventually represent the right fraction

$$Q(a) \circ Q(s)^{-1} : M \rightarrow N.$$

Definition 6.2.18. We define a relation \sim on the set $A \times S$ like this:

$$(a_1, s_1) \sim (a_2, s_2)$$

if there exist $b_1, b_2 \in A$ s.t.

$$a_1 \circ b_1 = a_2 \circ b_2 \text{ and } s_1 \circ b_1 = s_2 \circ b_2 \in S.$$

Note that the relation \sim imposes condition (ii) of Lemma 6.2.5.

Here it is in a commutative diagram, in which we have made the objects explicit:

$$(6.2.19) \quad \begin{array}{ccccc} & & K & & \\ & b_1 \swarrow & & \searrow b_2 & \\ L_1 & & & & L_2 \\ s_1 \downarrow & & & & \downarrow a_2 \\ M & \xleftarrow{s_2} & & \xrightarrow{a_1} & N \end{array}$$

The arrows ending at M are in S .

Lemma 6.2.20. *If the right Ore condition holds then the relation \sim is an equivalence.*

Proof. Reflexivity: take $K := L$ and $b_i := 1_L : L \rightarrow L$. Symmetry is trivial.

Now to prove transitivity. Suppose we are given $(a_1, s_1) \sim (a_2, s_2)$ and $(a_2, s_2) \sim (a_3, s_3)$. So we have the solid part of the first diagram in Figure 2, and it is commutative. The arrows ending at M are in S .

By condition (RD1) applied to $K \rightarrow M \leftarrow J$ there are $t \in S$ and $d \in A$ s.t.

$$(s_3 \circ c_3) \circ d = (s_1 \circ b_1) \circ t.$$

Comparing arrows $I \rightarrow M$ in this diagram, we see that

$$s_2 \circ (b_2 \circ t) = s_1 \circ b_1 \circ t = s_3 \circ c_3 \circ d = s_2 \circ (c_2 \circ d).$$

By (RD2) there is $u \in S$ s.t.

$$(b_2 \circ t) \circ u = (c_2 \circ d) \circ u.$$

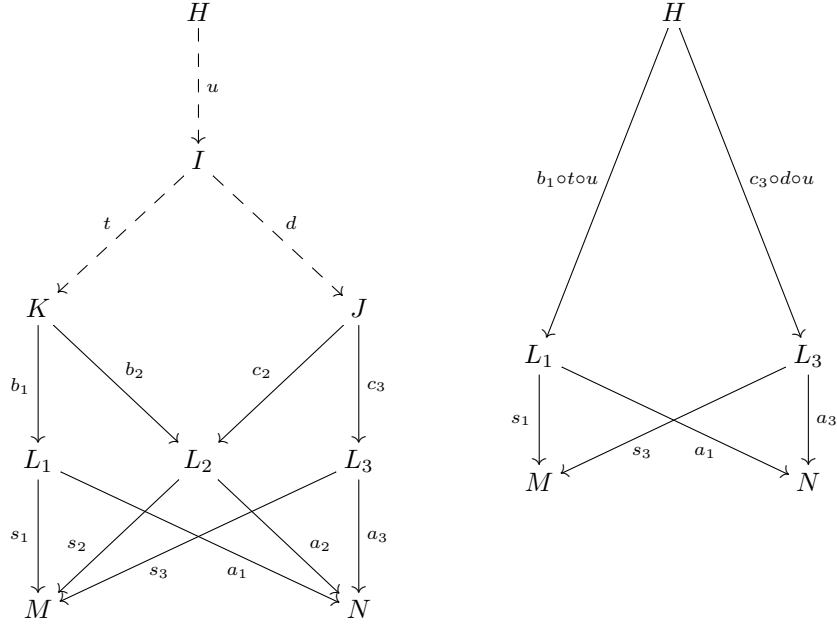


FIGURE 2.

So all paths $H \rightarrow M$ are equal and belong to \mathcal{S} , and all paths $H \rightarrow N$ are equal. Now delete the object L_2 and the arrows going through it. Then delete the objects I, J, K , but keep the paths going through them. We get the second diagram in Figure 2. It is commutative, and all arrows ending at M are in \mathcal{S} . This is evidence for $(a_1, s_1) \sim (a_3, s_3)$. \square

Proof of Theorem 6.2.15.

Step 1. In this step we prove (i) \Rightarrow (ii). Take $a \in \mathcal{A}$ and $s \in \mathcal{S}$. Consider $q := Q(s)^{-1} \circ Q(a)$. By (RO3) there are $b \in \mathcal{A}$ and $t \in \mathcal{S}$ s.t. $q = Q(b) \circ Q(t)^{-1}$. So

$$Q(s \circ b) = Q(a \circ t).$$

By (RO4) there is $u \in \mathcal{S}$ s.t.

$$(s \circ b) \circ u = (a \circ t) \circ u.$$

We read this as

$$s \circ (b \circ u) = a \circ (t \circ u),$$

and note that $t \circ u \in \mathcal{S}$. So (RD1) holds.

Next $a, b \in \mathcal{A}$ and $s \in \mathcal{S}$ s.t. $s \circ a = s \circ b$. Then $Q(s \circ a) = Q(s \circ b)$. But $Q(s)$ is invertible, so $Q(a) = Q(b)$. By (RO4) there is $t \in \mathcal{S}$ s.t. $a \circ t = b \circ t$. We have proved (RD2).

Step 2. Now we assume that condition (ii) holds, and we define the morphism sets $\mathcal{A}_\mathcal{S}(M, N)$, composition between them, and the identity morphisms.

For any $M, N \in \text{Ob}(\mathcal{A})$ let

$$\mathcal{A}_\mathcal{S}(M, N) := \frac{(\mathcal{A} \times \mathcal{S})(M, N)}{\sim},$$

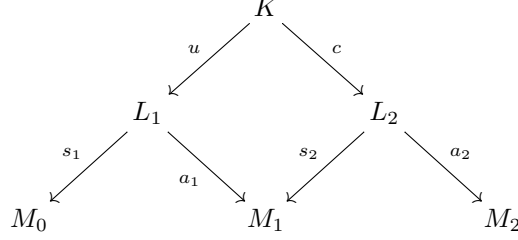


FIGURE 3.

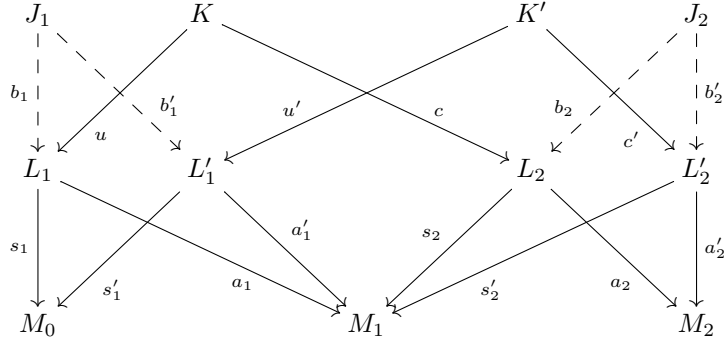


FIGURE 4.

where \sim is the relation from Definition 6.2.18, which is an equivalence relation by Lemma 6.2.20.

We define composition like this. Given $q_1 \in \mathbf{A}_S(M_0, M_1)$ and $q_2 \in \mathbf{A}_S(M_1, M_2)$, choose representatives $(a_i, s_i) \in (\mathbf{A} \times \mathbf{S})(M_{i-1}, M_i)$. We use the notation $q_i = \overline{(a_i, s_i)}$ to indicate this. By (RD1) there are $c \in \mathbf{A}$ and $u \in \mathbf{S}$ s.t. $s_2 \circ c = a_1 \circ u$. The composition

$$q_2 \circ q_1 \in \mathbf{A}_S(M_0, M_2)$$

is defined to be

$$q_2 \circ q_1 := \overline{(a_2 \circ c, s_1 \circ u)} \in (\mathbf{A} \times \mathbf{S})(M_0, M_2).$$

The idea behind the formula can be seen in the diagram in Figure 3.

We have to verify that this definition is independent of the representatives. So suppose we take other representatives $q_i = \overline{(a'_i, s'_i)}$, and we choose morphisms u', c' to construct the composition. This is the solid part of the diagram in Figure 4, and it is a commutative diagram. We must prove that

$$\overline{(a_2 \circ c, s_1 \circ u)} = \overline{(a'_2 \circ c', s'_1 \circ u')}.$$

There are morphisms b_i, b'_i the are evidence for $(a_i, s_i) \sim (a'_i, s'_i)$. They are depicted as the dashed arrows in Figure 4. That whole diagram is commutative. The morphisms $J_1 \rightarrow M_0$, $K \rightarrow M_0$, $K' \rightarrow M_0$ and $J_2 \rightarrow M_1$ are all in \mathbf{S} .

Choose $v_1 \in \mathbf{S}$ and $d_1 \in \mathbf{A}$ s.t. the first diagram in Figure 5 is commutative. This can be done by (RD1).

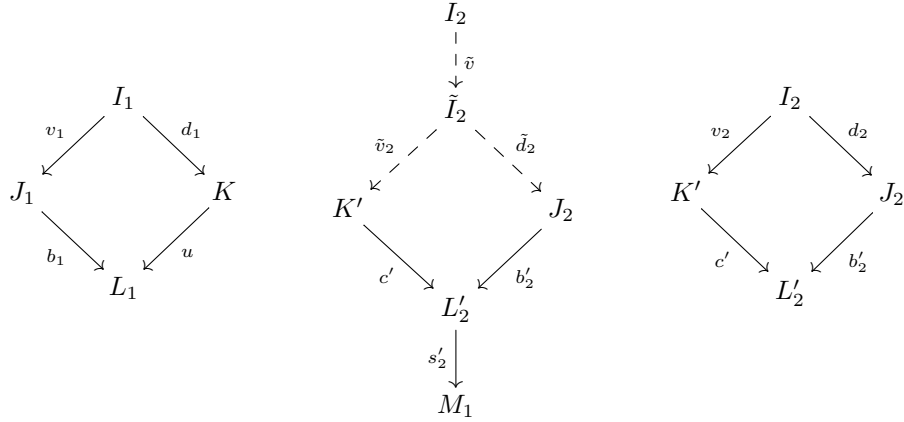


FIGURE 5.

Consider the solid part of the middle diagram in Figure 5. Since $J_2 \rightarrow M_1$ is in \mathbf{S} , by (RD1) there are $\tilde{v}_2 \in \mathbf{S}$ and $\tilde{d}_2 \in \mathbf{A}$ s.t. the two paths $\tilde{I}_2 \rightarrow M_1$ are equal. By (RD2) there is $\tilde{v} \in \mathbf{S}$ s.t. the two paths $I_2 \rightarrow L'_2$ are equal. We get the commutative diagram in the middle of Figure 5. Next, defining $d_2 := \tilde{d}_2 \circ \tilde{v}$ and $v_2 := \tilde{v}_2 \circ \tilde{v} \in \mathbf{S}$, we obtain the third commutative diagram in Figure 5.

We now embed the first and third diagrams from Figure 5 into the diagram in Figure 4. This gives us the solid diagram in Figure 6, and it is commutative. The morphisms $I_1 \rightarrow M_0$ belong to \mathbf{S} .

Choose $w \in \mathbf{S}$ and $e \in \mathbf{A}$, starting at an object H , to fill the diagram $I_1 \rightarrow M_0 \leftarrow I_2$, using (RD1). The path $H \rightarrow I_1 \rightarrow M_0$ is in \mathbf{S} , and all the paths $H \rightarrow M_0$ are equal. But we could have failure of commutativity in the paths $H \rightarrow L'_1$ and $H \rightarrow L_2$.

The two paths $H \rightarrow L'_1$ in Figure 6 satisfy

$$s'_1 \circ (b'_1 \circ v_1 \circ w) = s'_1 \circ (u' \circ v_2 \circ e).$$

Therefore there is $w' \in \mathbf{S}$ s.t.

$$(b'_1 \circ v_1 \circ w) \circ w' = (u' \circ v_2 \circ e) \circ w'.$$

Next, the two paths $H' \rightarrow L_2$ satisfy

$$s_2 \circ (c \circ d_1 \circ w \circ w') = s_2 \circ (b_2 \circ d_2 \circ e \circ w');$$

this is because we can take a detour through L'_1 . Therefore there is $w'' \in \mathbf{S}$ s.t.

$$(c \circ d_1 \circ w \circ w') \circ w'' = (b_2 \circ d_2 \circ e \circ w') \circ w''.$$

Now all paths $H'' \rightarrow M_2$ in Figure 6 are equal. All paths $H'' \rightarrow M_0$ are equal and are in \mathbf{S} .

Erase the objects M_1, J_1, J_2 and all arrows touching them from Figure 6. Then erase H, H' , but keep the paths through them. We obtain the commutative diagram in Figure 7. This is evidence for

$$(a_2 \circ c, s_1 \circ u) \sim (a'_2 \circ c', s'_1 \circ u').$$

The proof that composition is well-defined is done.

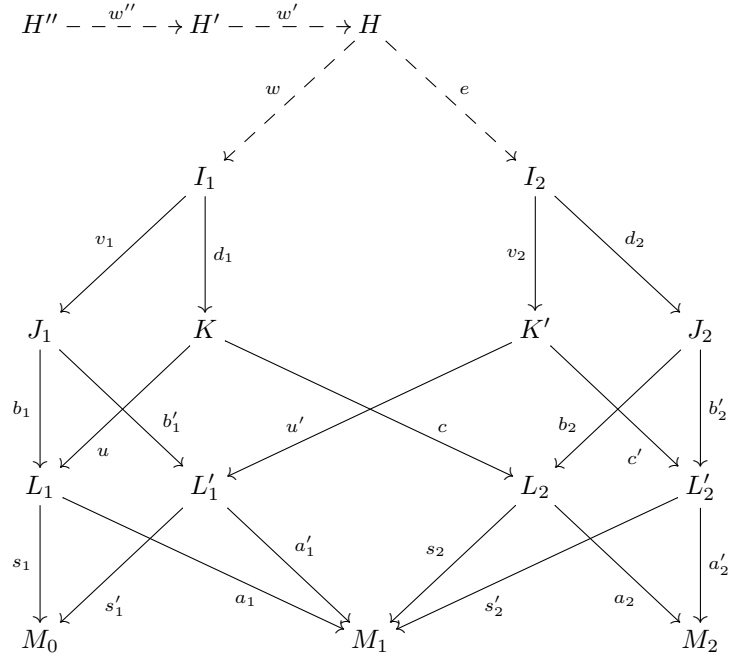


FIGURE 6.

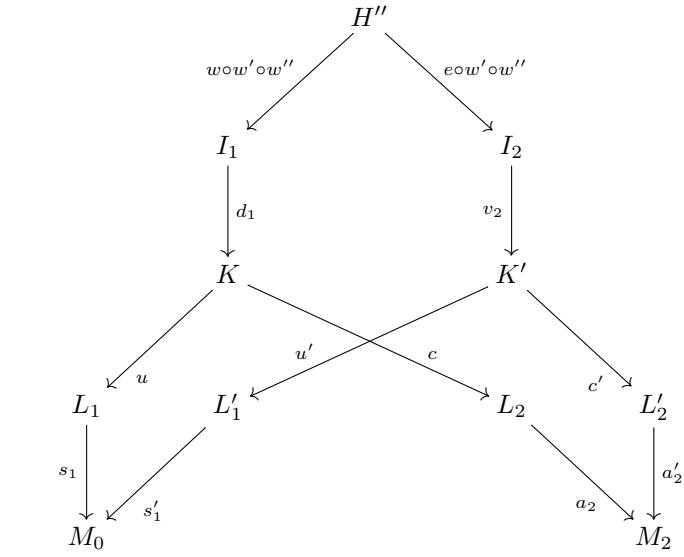


FIGURE 7.

The identity morphism 1_M of an object M is $\overline{(1_M, 1_M)}$.

Step 3. We have to verify the associativity and the identity properties of composition in \mathbf{A}_S . Namely that \mathbf{A}_S is a category. This seems to be not too hard, given Step 2, and we leave it as an exercise!

Step 4. The functor $Q : \mathbf{A} \rightarrow \mathbf{A}_S$ is defined to be $Q(M) := M$ on objects, and $Q(a) := \overline{(a, 1_M)}$ for $a : M \rightarrow N$ in \mathbf{A} . We have to verify this is a functor... Again, an exercise.

Step 5. Finally we verify properties (RO1)-(RO4). (RO1) is clear. The inverse of $Q(s)$ is $\overline{(1, s)}$, so (RO2) holds.

It is not hard to see that

$$\overline{(a, s)} = \overline{(a, 1)} \circ \overline{(1, s)};$$

this is (RO3).

If $Q(a_1) = Q(a_2)$, then $(a_1, 1_M) \sim (a_2, 1_M)$; so there are $b_1, b_2 \in \mathbf{A}$ s.t. $a_1 \circ b_1 = a_2 \circ b_2$ and $1 \circ b_1 = 1 \circ b_2 \in S$. Writing $s := b_1 \in S$, we get $a_1 \circ s = a_2 \circ s$. This proves (RO4). \square

Proposition 6.2.21. *Let \mathbf{A} be a category, let S be a right denominator set in \mathbf{A} , and let (\mathbf{A}_S, Q) be the right Ore localization. For any two morphisms $q_1, q_2 : M \rightarrow N$ in \mathbf{A}_S there is a common denominator. Namely we can write*

$$q_i = Q(a_i) \circ Q(s)^{-1}$$

for suitable $a_i \in \mathbf{A}$ and $s \in S$.

Proof. Choose representatives $q_i = Q(a'_i) \circ Q(s'_i)^{-1}$. By (RD1) applied to $L_1 \rightarrow M \leftarrow L_2$, there are $b \in \mathbf{A}$ and $t \in S$ s.t. the diagram above M commutes:

$$\begin{array}{ccc} & L & \\ t \swarrow & & \searrow b \\ L_1 & & L_2 \\ s'_1 \downarrow & \swarrow s'_2 & \searrow a'_1 \\ M & & N \end{array}$$

Write $s := s'_1 \circ t = s'_2 \circ b$, $a_1 := a'_1 \circ t$ and $a_2 := a'_2 \circ b$. By Lemma 6.2.5 we get $q_i = Q(a_i) \circ Q(s)^{-1}$. \square

Exercise 6.2.22. Let \mathbf{A} be a category, let S be a right denominator set in \mathbf{A} . Let Y be a subset of $\text{Ob}(\mathbf{A})$, and let \mathbf{B} and \mathbf{T} be the full subcategories of \mathbf{A} and S respectively on the set of objects Y .

- (1) Is \mathbf{T} a right denominator set in \mathbf{B} ?
- (2) Show that if \mathbf{T} is a right denominator set in \mathbf{B} , then the inclusion functor $F : \mathbf{B} \rightarrow \mathbf{A}$ extends uniquely to a functor $F_{\mathbf{T}} : \mathbf{B}_{\mathbf{T}} \rightarrow \mathbf{A}_S$.
- (3) Assume that \mathbf{T} is a right denominator set in \mathbf{B} . Is the functor $F_{\mathbf{T}}$ full or faithful?

We will return to these questions later.

6.3. Localization of Linear Categories. Until now in this section we dealt with arbitrary categories. In this and the subsequent subsection, our categories will be linear over some commutative base ring \mathbb{K} (that will be implicit in everything). This includes the case $\mathbb{K} = \mathbb{Z}$ of course.

For convenience we only talk about right denominator sets here. All the statements hold equally for left denominator sets; cf. Remark 6.2.2 and Proposition 6.2.3.

Theorem 6.3.1. *Let \mathbf{A} be a \mathbb{K} -linear category, let S be a right denominator set in \mathbf{A} , and let (\mathbf{A}_S, Q) be the right Ore localization.*

- (1) *The category \mathbf{A}_S has a unique \mathbb{K} -linear structure, such that $Q : \mathbf{A} \rightarrow \mathbf{A}_S$ is a \mathbb{K} -linear functor.*
- (2) *Suppose \mathbf{B} is another \mathbb{K} -linear category, and $F : \mathbf{A} \rightarrow \mathbf{B}$ is a \mathbb{K} -linear functor s.t. $F(s)$ is invertible for every $s \in S$. Let $F_S : \mathbf{A}_S \rightarrow \mathbf{B}$ be the localization of F . Then F_S is a \mathbb{K} -linear functor.*
- (3) *If \mathbf{A} is an additive category, then so is \mathbf{A}_S .*

Proof. (1) Let $q_i : M \rightarrow N$ be morphisms in \mathbf{A}_S . Choose common denominator presentations $q_i = Q(a_i) \circ Q(s)^{-1}$. Since Q must be an additive functor, we have to define

$$(6.3.2) \quad Q(a_1) + Q(a_2) := Q(a_1 + a_2).$$

By the distributive law (bilinearity of composition) we must define

$$q_1 + q_2 := Q(a_1 + a_2) \circ Q(s)^{-1}.$$

For $\lambda \in \mathbb{K}$ we must define

$$\lambda \cdot q_i := Q(\lambda \cdot a_i) \circ Q(s)^{-1}.$$

The usual tricks are then used to prove independence of representatives. For instance, to prove that (6.3.2) is independent of choices, suppose that $Q(a_1) = Q(a'_1)$ and $Q(a_2) = Q(a'_2)$. Then, by (RO4), there are $t_1, t_2 \in S$ such that $a_1 \circ t_1 = a'_1 \circ t_1$ and $a_2 \circ t_2 = a'_2 \circ t_2$. By (RD1) there exist $b \in \mathbf{A}$ and $v \in S$ s.t. $t_1 \circ b = t_2 \circ v$. Let $t_3 := t_2 \circ v \in S$. Then

$$(a_1 + a_2) \circ t_3 = (a'_1 + a'_2) \circ t_3,$$

and hence

$$Q(a_1 + a_2) = Q(a'_1 + a'_2).$$

In this way \mathbf{A}_S is a \mathbb{K} -linear category, and Q is a \mathbb{K} -linear functor.

(2) The only option for F_S is $F_S(q_i) := F(a_i) \circ F(s)^{-1}$. The usual tricks are used to prove independence of representatives.

(3) Clear from Propositions 2.4.2 and 2.4.5. □

Example 6.3.3. Let A be a ring, which we can think of as a one object linear category \mathbf{A} . In this context, Theorem 6.3.1 is one of the most important results in ring theory. See [Row, Ste].

Example 6.3.4. Suppose A is a commutative ring, and S is a multiplicatively closed set in it. Because A is commutative, the denominator conditions hold automatically. The localized category \mathbf{A}_S is the single object category, with endomorphism set A_S . This is simply the usual commutative localization.

Note that if S contains a nilpotent element, then the ring A_S is trivial.

The observation above should serve as a warning: localization can sometimes kill everything. This is the singularity effect: dividing by zero!

Fortunately, the localization procedure (7.0.11), that gives rise to the derived category, does not cause any catastrophe, as we shall see in Proposition 6.4.10.

Remark 6.3.5. Suppose A is a ring and S is a right denominator set in it. Then the right Ore localization A_S is *flat* as left A -module. See [Row, Theorem 3.1.20]. I have no idea if something like this is true for linear categories with more than one object.

Proposition 6.3.6. *Let (\mathbf{K}, T) be a T -additive \mathbb{K} -linear category, let S be a right denominator set in \mathbf{K} such that $T(S) = S$, and let $Q : \mathbf{K} \rightarrow \mathbf{K}_S$ be the localization functor.*

- (1) *There is a unique \mathbb{K} -linear automorphism T_S of the category \mathbf{K}_S , such that*

$$T_S \circ Q = Q \circ T$$

as functors $\mathbf{K} \rightarrow \mathbf{K}_S$.

- (2) *Let τ be the identity automorphism of the functor $Q \circ T$. Then*

$$(Q, \tau) : (\mathbf{K}, T) \rightarrow (\mathbf{K}_S, T_S)$$

is a T -additive functor.

Proof. (1) By the assumption the functor $Q \circ T : \mathbf{K} \rightarrow \mathbf{K}_S$ sends the morphisms in S to isomorphism. By the property (L3) of localization in Definition 6.1.2, the functor $T_S : \mathbf{K}_S \rightarrow \mathbf{K}_S$ satisfying $T_S \circ Q = Q \circ T$ exists and is unique. Similarly, there is a unique functor $T_S^{-1} : \mathbf{K}_S \rightarrow \mathbf{K}_S$ satisfying $T_S^{-1} \circ Q = Q \circ T^{-1}$. An easy calculation shows that

$$T_S^{-1} \circ T_S = \text{Id} = T_S \circ T_S^{-1}.$$

Hence T_S is an automorphism of \mathbf{K}_S .

- (2) This is clear. □

The composition of T -additive functors was defined in Definition 5.1.4.

Proposition 6.3.7. *In the situation of Proposition 6.3.6, suppose (\mathbf{K}', T') is another T -additive \mathbb{K} -linear category, and*

$$(F, \nu) : (\mathbf{K}, T) \rightarrow (\mathbf{K}', T')$$

is a T -additive \mathbb{K} -linear functor, such that $F(s)$ is invertible for any $s \in S$. Let $F_S : \mathbf{K}_S \rightarrow \mathbf{K}'$ be the localized functor. Then there is a unique isomorphism

$$\nu_S : F_S \circ T_S \xrightarrow{\cong} T' \circ F_S$$

of functors $\mathbf{K}_S \rightarrow \mathbf{K}'$, such that

$$(F, \nu) = (F_S, \nu_S) \circ (Q, \tau)$$

as T -additive functors $(\mathbf{K}, T) \rightarrow (\mathbf{K}', T')$.

Exercise 6.3.8. Prove Proposition 6.3.7.

6.4. Localization of Pretriangulated Categories. Let \mathbf{K} be a pretriangulated category, with translation functor T .

Proposition 6.4.1. *Suppose $H : \mathbf{K} \rightarrow \mathbf{M}$ is a cohomological functor, where \mathbf{M} is some abelian category. Let*

$$\mathbf{S} := \{s \in \mathbf{K} \mid H(T^i(s)) \text{ is invertible for all } i \in \mathbb{Z}\}.$$

Then \mathbf{S} is a left and right denominator set in \mathbf{K} .

Proof. It is clear that \mathbf{S} is closed under composition and contains the identity morphisms. So it is a multiplicatively closed set.

Let's prove that condition (RD1) of Definition 6.2.14 holds. Suppose we are given morphisms $L \xrightarrow{a} N \xleftarrow{s} M$ with $s \in \mathbf{S}$. We need to find morphisms $L \xleftarrow{t} K \xrightarrow{b} M$ with $t \in \mathbf{S}$ and such that $a \circ t = s \circ b$.

Consider the solid commutative diagram

$$\begin{array}{ccccccc} K & \xrightarrow{t} & L & \xrightarrow{coa} & P & \longrightarrow & T(K) \\ \downarrow b & & \downarrow a & & \downarrow = & & \downarrow T(b) \\ M & \xrightarrow{s} & N & \xrightarrow{c} & P & \longrightarrow & T(M) \end{array}$$

where the bottom row is a distinguished triangle built on $M \xrightarrow{s} N$, and the top row is a distinguished triangle built on $L \xrightarrow{coa} P$, then turned 120° to the right. By axiom (TR3) there is a morphism b making the diagram commutative. Thus $a \circ t = s \circ b$. Since $H(T^i(s))$ are invertible for all $i \in \mathbb{Z}$, it follows that $H(T^i(P)) = 0$. But then $H(T^i(t))$ are invertible for all $i \in \mathbb{Z}$, so $t \in \mathbf{S}$.

Next we prove condition (RD2) of Definition 6.2.14. Because we are in an additive category, this condition is simplified: given $a \in \mathbf{K}$ and $s \in \mathbf{S}$ satisfying $s \circ a = 0$, we have to find $t \in \mathbf{S}$ satisfying $a \circ t = 0$.

Say the objects involved are $L \xrightarrow{a} M \xrightarrow{s} N$. Take a distinguished triangle built on s and then turned:

$$P \xrightarrow{b} M \xrightarrow{s} N \rightarrow T(P).$$

We get an exact sequence

$$\mathrm{Hom}_{\mathbf{K}}(L, P) \xrightarrow{b \circ -} \mathrm{Hom}_{\mathbf{K}}(L, M) \xrightarrow{s \circ -} \mathrm{Hom}_{\mathbf{K}}(L, N).$$

Since $s \circ a = 0$, there is $c : L \rightarrow P$ s.t. $a = b \circ c$. Now look at a distinguished triangle built on c , and then turned:

$$K \xrightarrow{t} L \xrightarrow{c} P \rightarrow T(K).$$

We know that $c \circ t = 0$; hence $a \circ t = b \circ c \circ t = 0$. But $(s \in \mathbf{S}) \Rightarrow (H(T^i(P)) = 0 \text{ for all } i) \Rightarrow (t \in \mathbf{S})$.

The left versions (LD1) and (LD2) are proved the same way. \square

Definition 6.4.2. A *denominator set of cohomological origin* in \mathbf{K} is a denominator set $\mathbf{S} \subseteq \mathbf{K}$ that arises from a cohomological functor H , as in Proposition 6.4.1. The morphisms in \mathbf{S} are called *quasi-isomorphisms relative to H* .

Theorem 6.4.3. *Let (\mathbf{K}, T) be a pretriangulated category, let \mathbf{S} be a denominator set of cohomological origin in \mathbf{K} , and let*

$$(Q, \tau) : (\mathbf{K}, T) \rightarrow (\mathbf{K}_{\mathbf{S}}, T_{\mathbf{S}})$$

be the T -additive functor from Proposition 6.3.6. The T -additive category (K_S, T_S) has a unique pretriangulated structure such that these two properties hold:

- (i) The pair (Q, τ) is a triangulated functor.
- (ii) Suppose (K', T') is another pretriangulated category, and

$$(F, \nu) : (K, T) \rightarrow (K', T')$$

is a triangulated functor, such that $F(s)$ is invertible for every $s \in S$. Let

$$(F_S, \nu_S) : (K_S, T_S) \rightarrow (K', T')$$

be the T -additive functor from Proposition 6.3.7. Then (F_S, ν_S) is a triangulated functor.

Proof. Since S is of cohomological origin we have $T(S) = S$. Recall that the translation isomorphism τ is the identity automorphism of the functor $Q \circ T$; see Proposition 6.3.6. So we will ignore it.

Step 1. The distinguished triangles in K_S are defined to be those triangles that are isomorphic to the images under Q of distinguished triangles in K . Let us verify the axioms of pretriangulated category.

(TR1). By definition every triangle that's isomorphic to a distinguished triangle is distinguished; and the triangle

$$M \xrightarrow{1_M} M \rightarrow 0 \rightarrow T(M)$$

in K_S is clearly distinguished.

Suppose we are given a morphism $\alpha : L \rightarrow M$ in K_S . We have to build a distinguished triangle on it. Choose a fraction presentation $\alpha = Q(a) \circ Q(s)^{-1}$. Using condition (LD1) we can find $b \in K$ and $t \in S$ such that $t \circ a = b \circ s$. These fit into the solid commutative diagram

$$\begin{array}{ccccc} & & \alpha & & \\ & \swarrow & \text{---} & \searrow & \\ L & \xleftarrow{s} & K & \xrightarrow{a} & M \\ & \searrow & & \swarrow & \\ & & \tilde{K} & & \end{array}$$

(The dashed arrow α is in K_S .)

in K . (The dashed arrow α is in K_S .)

Consider the solid commutative diagram below, where the rows are distinguished triangles built on a and b respectively.

$$(6.4.4) \quad \begin{array}{ccccccc} K & \xrightarrow{a} & M & \xrightarrow{e} & N & \xrightarrow{c} & T(K) \\ \downarrow s & & \downarrow t & & \downarrow u & & \downarrow T(s) \\ L & \xrightarrow{b} & \tilde{K} & \longrightarrow & P & \xrightarrow{d} & T(L) \end{array}$$

By (TR3) there is a morphism u that makes the whole diagram commutative. Since $s, t \in S$ and H is a cohomological functor, it follows that $u \in S$. Applying the functor Q to (6.4.4), and using the isomorphism $Q(t) : M \rightarrow \tilde{K}$ to replace \tilde{K} with

M , we get the commutative diagram

$$\begin{array}{ccccccc}
 K & \xrightarrow{Q(a)} & M & \xrightarrow{Q(e)} & N & \xrightarrow{Q(c)} & T(K) \\
 Q(s) \downarrow & & Q(1_M) \downarrow & & Q(u) \downarrow & & T(Q(s)) \downarrow \\
 L & \xrightarrow{\alpha} & M & \xrightarrow{Q(u \circ e)} & P & \xrightarrow{Q(d)} & T(L)
 \end{array}$$

in K_S . The top row is a distinguished triangle, and the vertical arrows are isomorphisms. So the bottom row is a distinguished triangle. This is the triangle we were looking for.

(TR2). Turning: this is trivial.

(TR3). We are given the solid commutative diagram in K_S , where the rows are distinguished triangles:

$$(6.4.5) \quad \begin{array}{ccccccc}
 L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\
 \phi \downarrow & & \psi \downarrow & & \chi \downarrow & & T(\phi) \downarrow \\
 L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L')
 \end{array}$$

and we have to find χ to complete the diagram.

By replacing the rows with isomorphic triangles, we can assume they come from K . Thus we can replace (6.4.5) with this diagram:

$$(6.4.6) \quad \begin{array}{ccccccc}
 L & \xrightarrow{Q(\alpha)} & M & \xrightarrow{Q(\beta)} & N & \xrightarrow{Q(\gamma)} & T(L) \\
 \phi \downarrow & & \psi \downarrow & & \chi \downarrow & & T(\phi) \downarrow \\
 L' & \xrightarrow{Q(\alpha')} & M' & \xrightarrow{Q(\beta')} & N' & \xrightarrow{Q(\gamma')} & T(L')
 \end{array}$$

in which $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are morphisms in K . It is a commutative diagram. Let us choose fraction presentations $\phi = Q(a) \circ Q(s)^{-1}$ and $\psi = Q(b) \circ Q(t)^{-1}$. Then the solid diagram (6.4.6) comes from applying Q to the diagram

$$(6.4.7) \quad \begin{array}{ccccccc}
 L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\
 s \uparrow & & t \uparrow & & & & T(s) \uparrow \\
 \tilde{L} & & \tilde{M} & & & & T(\tilde{L}) \\
 a \downarrow & & b \downarrow & & & & T(a) \downarrow \\
 L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L')
 \end{array}$$

in K . Here the rows are distinguished triangles in K ; but the diagram might fail to be commutative.

By axiom (RO3) we can find $c \in K$ and $u \in S$ s.t.

$$Q(t)^{-1} \circ Q(\alpha) \circ Q(s) = Q(c) \circ Q(u)^{-1}.$$

This is the solid diagram:

$$\begin{array}{ccccc}
 & & L & \xrightarrow{\alpha} & M \\
 & & \uparrow s & & \uparrow t \\
 \tilde{L}'' & \xrightarrow{u'} & \tilde{L}' & \xrightarrow{u} & \tilde{L} & \xrightarrow{a} & \tilde{M} \\
 & & \downarrow c & \nearrow & \downarrow b \\
 & & L' & \xrightarrow{\alpha'} & M'
 \end{array}$$

Thus

$$Q(\alpha \circ s \circ u) = Q(t \circ c).$$

By (RO4) there is $u' \in \mathcal{S}$ s.t.

$$(\alpha \circ s \circ u) \circ u' = (t \circ c) \circ u'.$$

We get

$$\phi = Q(a) \circ Q(s)^{-1} = Q(a \circ u \circ u') \circ Q(s \circ u \circ u')^{-1}$$

in $\mathbf{K}_{\mathcal{S}}$. Thus, after substituting $\tilde{L} := \tilde{L}'$, $s := s \circ u \circ u'$, $a := a \circ u \circ u'$ and $c := c \circ u'$, we get a new diagram

$$\begin{array}{ccccccc}
 L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\
 \uparrow s & & \uparrow t & & & & \uparrow T(s) \\
 \tilde{L} & \xrightarrow{c} & \tilde{M} & & & & T(\tilde{L}) \\
 \downarrow a & & \downarrow b & & & & \downarrow T(a) \\
 L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L')
 \end{array}
 \tag{6.4.8}$$

in \mathbf{K} instead of (6.4.7). In this new diagram the top left square is commutative; but maybe the bottom left square is not commutative.

When we apply Q to the diagram (6.4.8), the whole diagram, including the bottom left square, becomes commutative, since (6.4.6) is commutative. Again using condition (RO4), there is $v \in \mathcal{S}$ s.t.

$$(\alpha' \circ a) \circ v = (b \circ c) \circ v.$$

In a diagram:

$$\begin{array}{ccccc}
 & & L & \xrightarrow{\alpha} & M \\
 & & \uparrow s & & \uparrow t \\
 \tilde{L}' & \xrightarrow{v} & \tilde{L} & \xrightarrow{c} & \tilde{M} \\
 & & \downarrow a & & \downarrow b \\
 & & L' & \xrightarrow{\alpha'} & M'
 \end{array}$$

Performing the replacements $\tilde{L} := \tilde{L}'$, $s := s \circ v$, $c := c \circ v$ and $a := a \circ v$ we now have a commutative square also at the bottom left of (6.4.8). Since $\gamma \circ \beta = 0$ and $\gamma' \circ \beta' = 0$, in fact the whole diagram (6.4.8) in \mathbf{K} is now commutative.

Now by (TR1) we can embed the morphism c in a distinguished triangle. We get the solid diagram

$$(6.4.9) \quad \begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & T(L) \\ \uparrow s & & \uparrow t & & \uparrow w & & \uparrow T(s) \\ \tilde{L} & \xrightarrow{c} & \tilde{M} & \xrightarrow{\tilde{\beta}} & \tilde{N} & \xrightarrow{\tilde{\gamma}} & T(\tilde{L}) \\ \downarrow a & & \downarrow b & & \downarrow d & & \downarrow T(a) \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & T(L') \end{array}$$

in \mathbf{K} . The rows are distinguished triangles. Since $\tilde{\gamma} \circ \tilde{\beta} = 0$, the solid diagram is commutative. By (TR3) there are morphisms w and d that make the whole diagram commutative. Now the morphism $w \in \mathbf{S}$ by the usual long exact sequence argument. The morphism

$$\chi := Q(d) \circ Q(w)^{-1} : N \rightarrow N'$$

solves the problem.

Step 2. Suppose (F, ν) is a triangulated functor as in condition (ii). By Proposition 6.3.7 this extends uniquely to a T -additive functor (F_S, ν_S) . The construction of the pretriangulated structure on (\mathbf{K}_S, T_S) in the previous steps, and the defining property of the translation isomorphism ν_S in Proposition 6.3.7, show that (F_S, ν_S) is a triangulated functor.

Step 3. At this point (\mathbf{K}_S, T_S) is a pretriangulated category, and conditions (i)-(ii) of the theorem are satisfied. We need to prove the uniqueness of the pretriangulated structure on (\mathbf{K}_S, T_S) . Condition (i) says that we can't have less distinguished triangles than those we declared. We can't have more distinguished triangles, because of condition (ii). \square

Proposition 6.4.10. *Consider the situation of Proposition 6.4.1 and Theorem 6.4.3.*

- (1) *The cohomological functor $H : \mathbf{K} \rightarrow \mathbf{M}$ factors into $H = H_S \circ Q$, where $H_S : \mathbf{K}_S \rightarrow \mathbf{M}$ is a cohomological functor.*
- (2) *Let M be an object of \mathbf{K} . The object $Q(M)$ is zero in \mathbf{K}_S iff the objects $H(T^i(M))$ are zero in \mathbf{M} for all i .*

Proof. (1) The existence and uniqueness of the functor H_S are by the universal property (L3) in Definition 6.1.2. We leave it as an exercise to show that H_S is a cohomological functor.

(2) Since H_S is an additive functor, if $Q(M) = 0$, then so is $H(M) = H_S(Q(M))$. And of course $Q(M) = 0$ iff $Q(T^i(M)) = 0$ for all i .

For the converse, let $\phi : 0 \rightarrow M$ be the zero morphism in \mathbf{K} . If $H(T^i(M)) = 0$ for all i , then $H(T^i(\phi)) : 0 \rightarrow H(T^i(M))$ are isomorphisms for all i . Then $\phi \in \mathbf{S}$, and so $Q(\phi) : 0 \rightarrow Q(M)$ is an isomorphism in \mathbf{K}_S . \square

7. THE DERIVED CATEGORY $\mathbf{D}(A, \mathbf{M})$

In this section there is a commutative base ring \mathbb{K} , that shall remain implicit most of the time. We fix a central DG \mathbb{K} -ring A , and a \mathbb{K} -linear abelian category \mathbf{M} . The DG category $\mathbf{C}(A, \mathbf{M})$ was introduced in Subsection 3.7, and the pretriangulated category $\mathbf{K}(A, \mathbf{M})$ was introduced in Subsection 5.4.

The functor $H^0 : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{M}$ is a cohomological functor, in the sense of Definition 5.3.2. The resulting denominator set is denoted by $\mathbf{S}(A, \mathbf{M})$, and its elements are called *quasi-isomorphisms*. The *derived category* of (A, \mathbf{M}) is the pretriangulated category

$$(7.0.11) \quad \mathbf{D}(A, \mathbf{M}) := \mathbf{K}(A, \mathbf{M})_{\mathbf{S}(A, \mathbf{M})}.$$

7.1. Definition of the Derived Category.

Proposition 7.1.1. *Let \mathbf{M} be an abelian category and let A be a DG ring. The functor*

$$H^0 : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{M}$$

is cohomological.

Proof. Clearly H^0 is additive. Consider a distinguished triangle

$$(7.1.2) \quad L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

in $\mathbf{K}(A, \mathbf{M})$. We can assume that it is the image of a standard triangle in $\mathbf{C}(A, \mathbf{M})$, namely that N is the cone associated to α , as in Definition 3.9.5, $\beta = e_\alpha$ and $\gamma = p_\alpha$. By construction, the cone N sits in an exact sequence of complexes

$$(7.1.3) \quad 0 \rightarrow M \xrightarrow{e_\alpha} N \xrightarrow{p_\alpha} T(L) \rightarrow 0.$$

Consider the diagram

$$\begin{array}{ccccc} H^{-1}(T(L)) & \xrightarrow{\text{conn}} & H^0(M) & \xrightarrow{H^0(e_\alpha)} & H^0(N) \\ \downarrow H(t_L^{-1}) & & \downarrow = & & \downarrow = \\ H^0(L) & \xrightarrow{H^0(\alpha)} & H^0(M) & \xrightarrow{H^0(\beta)} & H^0(N) \end{array}$$

in \mathbf{M} , where the first row is part of the long exact cohomology sequence for (7.1.3), and the second row comes from (7.1.2). The first square is commutative because any lifting represents the connecting homomorphism (cf. [Rot, Theorem 6.2]). The second square is also commutative. It follows that the diagram is commutative, and that the bottom row is exact. \square

Definition 7.1.4. A morphism ϕ in $\mathbf{K}(A, \mathbf{M})$ is called a *quasi-isomorphism* if the morphisms $H^i(\phi)$ in \mathbf{M} are isomorphisms for all i .

The set of quasi-isomorphisms in $\mathbf{K}(A, \mathbf{M})$ is denoted by $\mathbf{S}(A, \mathbf{M})$.

Note that $H^i = H^0 \circ T^i$. By Proposition 7.1.1 the functor H^0 is cohomological. Therefore $\mathbf{S}(A, \mathbf{M})$ is a denominator set of cohomological origin, Theorem 6.4.3 applies to it, and the next definition makes sense.

Definition 7.1.5. Let \mathbf{M} be a \mathbb{K} -linear abelian category and A a central DG \mathbb{K} -ring. The *derived category* of (A, \mathbf{M}) is the \mathbb{K} -linear pretriangulated category

$$\mathbf{D}(A, \mathbf{M}) := \mathbf{K}(A, \mathbf{M})_{\mathbf{S}(A, \mathbf{M})}.$$

In our situation we have additive functors

$$\mathbf{C}_{\text{str}}(A, \mathbf{M}) \xrightarrow{P} \mathbf{K}(A, \mathbf{M}) \xrightarrow{Q} \mathbf{D}(A, \mathbf{M}),$$

that are the identity on objects. Recall that the functor P sends a strict morphism of DG modules to its homotopy class; and Q is the localization functor with respect to quasi-isomorphisms.

Definition 7.1.6. Let \mathbf{M} be an abelian category and let A be a DG ring. Define the functor

$$\tilde{Q} := Q \circ P : \mathbf{C}_{\text{str}}(A, \mathbf{M}) \rightarrow \mathbf{D}(A, \mathbf{M}).$$

This definition will only be used in the present section.

It is sometimes convenient to describe morphisms in $\mathbf{D}(A, \mathbf{M})$ in terms of the functor \tilde{Q} . A morphism $s \in \mathbf{C}_{\text{str}}(A, \mathbf{M})$ is called a quasi-isomorphism if $P(s)$ is a quasi-isomorphism in $\mathbf{K}(A, \mathbf{M})$; i.e. if all the $H^i(s)$ are isomorphisms.

Proposition 7.1.7.

- (1) Any morphism ϕ in $\mathbf{D}(A, \mathbf{M})$ can be written as a right fraction

$$\phi = \tilde{Q}(a) \circ \tilde{Q}(s)^{-1}$$

where $a, s \in \mathbf{C}_{\text{str}}(A, \mathbf{M})$ and s is a quasi-isomorphism.

- (2) The kernel of \tilde{Q} is this: $\tilde{Q}(a) = 0$ in $\mathbf{D}(A, \mathbf{M})$ iff there exists a quasi-isomorphism s in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ such that $a \circ s$ is a coboundary in $\mathbf{C}(A, \mathbf{M})$.

Proof. (1) This is because of property (RO3) of Definition 6.2.1 and the fact that P is full.

(2) Property (RO4) of Definition 6.2.1 tells us what the kernel of Q is; and by definition the kernel of P is the 0-coboundaries. \square

Of course there is a left version of this proposition.

Exercise 7.1.8. Here $\mathbf{M} = \text{Mod } \mathbb{K}$, so $\mathbf{K}(A, \mathbf{M}) = \mathbf{K}(A)$. Show that the functor $H^0 : \mathbf{K}(A) \rightarrow \text{Mod } \mathbb{K}$ is corepresentable by the object $A \in \mathbf{K}(A)$ (see Subsection 1.7).

7.2. Localization of Subcategories of $\mathbf{K}(A, \mathbf{M})$.

Definition 7.2.1. Let \mathbf{K} be a pretriangulated category. A *full pretriangulated subcategory* of \mathbf{K} is a subcategory $\mathbf{L} \subseteq \mathbf{K}$ satisfying these conditions:

- (a) \mathbf{L} is a full additive subcategory (see Definition 2.2.5).
- (b) \mathbf{L} is closed under translations, i.e. $L \in \mathbf{L}$ iff $T(L) \in \mathbf{L}$.
- (c) \mathbf{L} is closed under distinguished triangles, i.e. if

$$L' \rightarrow L \rightarrow L'' \rightarrow T(L)$$

is a distinguished triangle in \mathbf{K} s.t. $L', L \in \mathbf{L}$, then also $L'' \in \mathbf{L}$.

Observe that \mathbf{L} itself is pretriangulated, and the inclusion $\mathbf{L} \rightarrow \mathbf{K}$ is a triangulated functor.

Denominator sets of cohomological origin were introduced in Definition 6.4.2. By Theorem 6.4.3, if $\mathbf{S} \subseteq \mathbf{K}$ is a denominator set of cohomological origin, then the localization $\mathbf{K}_{\mathbf{S}}$ is a pretriangulated category.

Example 7.2.2. This is the most important example for us: $\mathbf{K} = \mathbf{K}(A, \mathbf{M})$, $H = H^0 : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{M}$ and $\mathbf{S} = \mathbf{S}(A, \mathbf{M})$. Here $\mathbf{K}_{\mathbf{S}} = \mathbf{D}(A, \mathbf{M})$, the derived category.

Proposition 7.2.3. *Let K be a pretriangulated category, let S be a denominator set of cohomological origin in K , and let K' be a full pretriangulated subcategory of K . Then $S' := K' \cap S$ is a denominator set of cohomological origin in K' , the Ore localization $K'_{S'}$ exists, and $K'_{S'}$ is a pretriangulated category.*

Proof. Let $H : K \rightarrow M$ be a cohomological functor that determines S . The functor $H|_{K'} : K' \rightarrow M$ is also cohomological, and the set of morphisms S' satisfies

$$S' = \{s \in K' \mid H|_{K'}(T^i(s)) \text{ is an isomorphism for all } i\}.$$

Hence Proposition 6.4.1 and Theorem 6.4.3 apply. \square

In the situation of the proposition, the localization functor is denoted by $Q' : K' \rightarrow K'_{S'}$.

Proposition 7.2.4. *In the situation of Proposition 7.2.3, let $F : K' \rightarrow E$ be a triangulated functor into some pretriangulated category E . Assume that for every $s \in S'$, the morphism $F(s)$ is an isomorphism in E . Then there is a unique triangulated functor $F_{S'} : K'_{S'} \rightarrow E$ that extends F ; namely $F_{S'} \circ Q' = F$ as functors $K' \rightarrow E$.*

Proof. This is part of Theorem 6.4.3. \square

In particular we can look at the functor $F : K' \xrightarrow{\text{inc}} K \xrightarrow{Q} K_S$, and its extension $F_{S'} : K'_{S'} \rightarrow K_S$. We are interested in sufficient conditions for the functor $F_{S'}$ to be fully faithful.

Proposition 7.2.5. *Let K be a pretriangulated category, let S be a denominator set of cohomological origin in K , and let $K' \subseteq K$ be a full pretriangulated subcategory. Define $S' := K' \cap S$. Assume either of these conditions holds:*

- (r) *Let $M \in \text{Ob}(K)$. If there exists a morphism $s : M \rightarrow L$ in S with $L \in \text{Ob}(K')$, there exists a morphism $t : K \rightarrow M$ in S with $K \in \text{Ob}(K')$.*
- (l) *The same, but with arrows reversed.*

Then the functor $F_{S'} : K'_{S'} \rightarrow K_S$ is fully faithful.

Proof. We will prove the proposition under condition (r); the other condition is done the same way.

Let $L_1, L_2 \in \text{Ob}(K')$, and let $q : L_1 \rightarrow L_2$ be a morphism in K_S . Choose a presentation $q = Q(a) \circ Q(s)^{-1}$ with $s : M \rightarrow L_1$ a morphism in S and $a : M \rightarrow L_2$ a morphism in K . By condition (r) we can find a morphism $t : K \rightarrow M$ in S with $K \in \text{Ob}(K')$.

$$\begin{array}{ccc} & K & \\ & \downarrow t & \\ & M & \\ & \swarrow s \quad \searrow a & \\ L_1 & \xrightarrow{\quad q \quad} & L_2 \end{array}$$

Then $q = Q(a \circ t) \circ Q(s \circ t)^{-1}$. But $s \circ t \in S'$ and $a \circ t \in K'$, so q is in the image of the functor $F_{S'}$. We see that $F_{S'}$ is full.

Now let $q' : L_1 \rightarrow L_2$ be a morphism in $K'_{S'}$ such that $F_{S'}(q') = 0$. Let us denote the localization functor $K' \rightarrow K'_{S'}$ by Q' . Choose a presentation $q' = Q'(a) \circ Q'(s)^{-1}$,

with $s : N \rightarrow L_1$ a morphism in S' and $a : N \rightarrow L_2$ a morphism in K' . Because $F_{S'}(q') = 0$, and using Lemma 6.2.5, there is a morphism $u : M \rightarrow N$ in K such that $a \circ u = 0$ and $s \circ u \in S$. Note that $u \in S$. By condition (r), applied to $u : M \rightarrow N$, there is a morphism $t : K \rightarrow M$ in S such that $K \in \text{Ob}(K')$.

$$\begin{array}{ccc}
 & K & \\
 & \downarrow t & \\
 & M & \\
 & \downarrow u & \\
 & N & \\
 s \swarrow & & \searrow a \\
 L_1 & \xrightarrow{\quad q' \quad} & L_2
 \end{array}$$

Then we have

$$q' = Q'(a \circ u \circ t) \circ Q'(s \circ u \circ t)^{-1} = 0.$$

This proves that $F_{S'}$ is faithful. \square

7.3. Boundedness Conditions. A graded object $M = \{M^i\}_{i \in \mathbb{Z}}$ of M is said to be *bounded above* if the set $\{i \mid M^i \neq 0\}$ is bounded above. Likewise we define *bounded below* and *bounded* graded objects.

Definition 7.3.1. We define $\mathbf{C}^-(A, M)$, $\mathbf{C}^+(A, M)$ and $\mathbf{C}^b(A, M)$ to be full subcategories of $\mathbf{C}(A, M)$ consisting of bounded above, bounded below and bounded complexes respectively.

Likewise we define $\mathbf{K}^-(A, M)$, $\mathbf{K}^+(A, M)$ and $\mathbf{K}^b(A, M)$ to be the corresponding full subcategories of $\mathbf{K}(A, M)$.

Of course

$$\mathbf{C}^b(A, M) = \mathbf{C}^-(A, M) \cap \mathbf{C}^+(A, M),$$

and the same for $\mathbf{K}^b(A, M)$. The subcategories $\mathbf{K}^\star(A, M)$, for $\star \in \{-, +, b\}$, are full pretriangulated subcategory of $\mathbf{K}(A, M)$; this is because the operations of translation and cone preserve the various boundedness conditions.

As the next example shows, sometimes the category $\mathbf{K}^\star(A, M)$ can be very degenerate.

Example 7.3.2. Let A be the DG ring $\mathbb{K}[t, t^{-1}]$, the ring of Laurent polynomials in the variable t of degree 1, with the zero differential. If $M = \{M^i\}_{i \in \mathbb{Z}}$ is a nonzero object of $\mathbf{C}(A, M)$, then $M^i \neq 0$ for all i . Therefore the categories $\mathbf{C}^\star(A, M)$ and $\mathbf{K}^\star(A, M)$ are zero for $\star \in \{-, +, b\}$.

Let

$$\mathbf{S}^\star(A, M) := \mathbf{K}^\star(A, M) \cap \mathbf{S}(A, M),$$

the category of quasi-isomorphisms in $\mathbf{K}^\star(A, M)$. As already mentioned, Theorem 6.4.3 applies here, so we can localize.

Definition 7.3.3. For $\star \in \{-, +, b\}$ we define

$$\mathbf{D}^\star(A, M) := \mathbf{K}^\star(A, M)_{\mathbf{S}^\star(A, M)},$$

the Ore localization of $\mathbf{K}^\star(A, M)$ with respect to $\mathbf{S}^\star(A, M)$.

Here is another kind of boundedness condition.

Definition 7.3.4. For $\star \in \{-, +, b\}$ we define $\mathbf{D}(A, M)^\star$ to be the full subcategory of $\mathbf{D}(A, M)$ on the complexes M whose cohomology $H(M)$ is of boundedness type \star .

Of course $\mathbf{D}(A, M)^\star$ is a full pretriangulated subcategory of $\mathbf{D}(A, M)$.

The next proposition refers to the abelian case only – namely to $\mathbf{D}(M) = \mathbf{D}(\mathbb{K}, M)$. See Exercise 7.3.12 for a generalization to $\mathbf{D}(A, M)$ for a special sort of DG ring A .

Proposition 7.3.5. For $\star \in \{-, +, b\}$ the canonical functor $\mathbf{D}^\star(M) \rightarrow \mathbf{D}(M)^\star$ is an equivalence of pretriangulated categories.

Proof. Step 1. Here we prove that $F^- : \mathbf{D}^-(M) \rightarrow \mathbf{D}(M)$ is fully faithful. Let $s : M \rightarrow L$ be a quasi-isomorphism with $L \in \mathbf{K}^-(M)$. Say L is concentrated in degrees $\leq i$. Then $H^j(M) = H^j(L) = 0$ for all $j > i$. Consider the *smart truncation* of M at i :

$$(7.3.6) \quad \text{smt}^{\leq i}(M) := (\cdots \rightarrow M^{i-2} \xrightarrow{d} M^{i-1} \xrightarrow{d} Z^i(M) \rightarrow 0 \rightarrow \cdots)$$

where $Z^i(M) := \text{Ker}(d : M^i \rightarrow M^{i+1})$, the object of i -cocycles, is in degree i . Then $\text{smt}^{\leq i}(M)$ is a subcomplex of M , $\text{smt}^{\leq i}(M) \in \mathbf{K}^-(M)$, and the inclusion $t : \text{smt}^{\leq i}(M) \rightarrow M$ is a quasi-isomorphism. According to Proposition 7.2.5, with $\mathbf{K} = \mathbf{K}(M)$ and $\mathbf{K}' = \mathbf{K}^-(M)$, and with condition (r), we see that $F^- : \mathbf{D}^-(M) \rightarrow \mathbf{D}(M)$ is fully faithful.

Step 2. Here we prove that $F^+ : \mathbf{D}^+(M) \rightarrow \mathbf{D}(M)$ is fully faithful. Let $s : L \rightarrow M$ be a quasi-isomorphism with $L \in \mathbf{K}^+(M)$. Say L is concentrated in degrees $\geq i$. Then $H^j(M) = H^j(L) = 0$ for all $j < i$. Consider the other smart truncation of M at i :

$$(7.3.7) \quad \text{smt}^{\geq i}(M) := (\cdots \rightarrow 0 \rightarrow Y^i(M) \xrightarrow{d} M^{i+1} \xrightarrow{d} M^{i+2} \rightarrow \cdots)$$

where

$$(7.3.8) \quad Y^i(M) := \text{Coker}(d : M^{i-1} \rightarrow M^i)$$

is in degree i . Then $\text{smt}^{\geq i}(M)$ is a quotient complex of M , $\text{smt}^{\geq i}(M) \in \mathbf{K}^+(M)$, and the projection $t : M \rightarrow \text{smt}^{\geq i}(M)$ is a quasi-isomorphism. According to Proposition 7.2.5, with condition (l), we see that $F^+ : \mathbf{D}^+(M) \rightarrow \mathbf{D}(M)$ is fully faithful.

Step 3. The arguments in step 1 we show that $\mathbf{D}^b(M) \rightarrow \mathbf{D}^+(M)$ is fully faithful. And by step 2, $\mathbf{D}^+(M) \rightarrow \mathbf{D}(M)$ is fully faithful. Therefore $\mathbf{D}^b(M) \rightarrow \mathbf{D}(M)$ is fully faithful.

Step 4. Smart truncation shows that the functor $\mathbf{D}^\star(M) \rightarrow \mathbf{D}(M)^\star$ is essentially surjective on objects. \square

Remark 7.3.9. Most advanced texts write $\mathbf{D}^\star(M)$ instead of $\mathbf{D}(M)^\star$, and do not use the notation $\mathbf{D}(M)^\star$ at all. This is harmless by Proposition 7.3.5.

Remark 7.3.10. The object $Y^p(M) = \text{Coker}(d_M^{p-1})$ that appears in formula (7.3.8) does not have a name. The naming conventions would indicate that it should be called the “object of cococycles”, because it plays a role that’s dual to the role of the object of cocycles $Z^p(M) = \text{Ker}(d_M^p)$, and it can’t be called “cycles”. But the name “cococycles” sounds a bit strange.

Definition 7.3.11. A DG ring A is called *nonpositive* if $A^i = 0$ for all $i > 0$.

Exercise 7.3.12. Let A be a nonpositive DG ring and let \mathbf{M} be an abelian category.

- (1) Prove that differential on any $M \in \mathbf{C}_{\text{str}}(A, \mathbf{M})$ is A^0 -linear.
- (2) Prove that the smart truncations from formulas (7.3.6) and (7.3.8) are functors from $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ to itself.
- (3) Prove Proposition 7.3.5 for $\mathbf{C}_{\text{str}}(A, \mathbf{M})$.

7.4. Thick Subcategories of \mathbf{M} . Let \mathbf{M} be an abelian category. A *thick abelian subcategory* of \mathbf{M} is a full abelian subcategory \mathbf{N} that is closed under extensions. Namely if

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is a short exact sequence in \mathbf{M} with $M', M'' \in \mathbf{N}$, then $M \in \mathbf{N}$ too.

Let $\mathbf{D}_{\mathbf{N}}(\mathbf{M})$ be the full subcategory of $\mathbf{D}(\mathbf{M})$ consisting of complexes M such that $H^i(M) \in \mathbf{N}$ for every i .

Proposition 7.4.1. *If \mathbf{N} is a thick abelian subcategory of \mathbf{M} then $\mathbf{D}_{\mathbf{N}}(\mathbf{M})$ is a full pretriangulated subcategory of $\mathbf{D}(\mathbf{M})$.*

Proof. Clearly $\mathbf{D}_{\mathbf{N}}(\mathbf{M})$ is closed under translations. Now suppose

$$M' \rightarrow M \rightarrow M'' \rightarrow M[1]$$

is a distinguished triangle in $\mathbf{D}(\mathbf{M})$ such that $M', M \in \mathbf{D}_{\mathbf{N}}(\mathbf{M})$; we have to show that M'' is also in $\mathbf{D}_{\mathbf{N}}(\mathbf{M})$. Consider the exact sequence

$$H^i(M') \rightarrow H^i(M) \rightarrow H^i(M'') \rightarrow H^{i+1}(M') \rightarrow H^{i+1}(M).$$

The four outer objects belong to \mathbf{N} . Since \mathbf{N} is a thick abelian subcategory of \mathbf{M} it follows that $H^i(M'') \in \mathbf{N}$. \square

Example 7.4.2. Let A be a noetherian commutative ring. The category $\text{Mod}_f A$ of finitely generated modules is a thick abelian subcategory of $\text{Mod } A$.

Example 7.4.3. Consider $\text{Mod } \mathbb{Z} = \text{Ab}$. As above we have the thick abelian subcategory $\text{Ab}_{\text{fgen}} = \text{Mod}_f \mathbb{Z}$ of finitely generated abelian groups. There is also the thick abelian subcategory Ab_{tors} of torsion abelian groups (every element has a finite order). The intersection of Ab_{tors} and Ab_{fgen} is the category Ab_{fin} of finite abelian groups. This is also thick.

Example 7.4.4. Let X be a noetherian scheme (e.g. an algebraic variety over an algebraically closed field). Consider the abelian category $\text{Mod } \mathcal{O}_X$ of \mathcal{O}_X -modules. In it there is the thick abelian subcategory $\text{QCoh } \mathcal{O}_X$ of quasi-coherent sheaves, and in that there is the thick abelian subcategory $\text{Coh } \mathcal{O}_X$ of coherent sheaves.

For a left noetherian ring A we write

$$\mathbf{D}_f(\text{Mod } A) := \mathbf{D}_{\text{Mod } A}(\text{Mod } A).$$

Proposition 7.4.5. *Let A be a left noetherian ring and $\star \in \{-, b\}$. Then the canonical functor*

$$\mathbf{D}^{\star}(\text{Mod}_f A) \rightarrow \mathbf{D}_f(\text{Mod } A)^{\star}$$

is an equivalence of pretriangulated categories.

Proof. Consider the functor

$$F : \mathbf{D}^-(\mathrm{Mod}_f A) \rightarrow \mathbf{D}(\mathrm{Mod} A).$$

Suppose $s : M \rightarrow L$ is a quasi-isomorphism in $\mathbf{K}(\mathrm{Mod} A)$, such that $L \in \mathbf{K}^-(\mathrm{Mod}_f A)$. Then $M \in \mathbf{D}_f^-(\mathrm{Mod} A)$. A bit later (in Corollary 10.3.32) we will prove that M admits a free resolution $P \rightarrow M$, where P is a bounded above complex of finitely generated free modules. Thus we get a quasi-isomorphism $t : P \rightarrow M$ with $P \in \mathbf{K}^-(\mathrm{Mod}_f A)$. By Proposition 7.2.5 with condition (r) we conclude that F is fully faithful. This also shows that the essential image of F is $\mathbf{D}_f^-(\mathrm{Mod} A)$.

Next consider the functor

$$G : \mathbf{D}^b(\mathrm{Mod}_f A) \rightarrow \mathbf{D}^-(\mathrm{Mod}_f A).$$

Suppose $s : L \rightarrow M$ is a quasi-isomorphism in $\mathbf{K}^-(\mathrm{Mod}_f A)$ with $L \in \mathbf{K}^b(\mathrm{Mod}_f A)$. Say $H(L)$ is concentrated in the integer interval $[d_0, d_1]$. Then $t : M \rightarrow \mathrm{smt}^{\geq d_0}(M)$ is a quasi-isomorphism, and $\mathrm{smt}^{\geq d_0}(M) \in \mathbf{K}^b(\mathrm{Mod}_f A)$. By Proposition 7.2.5 with condition (l) we conclude that G is fully faithful. Therefore the composition

$$F \circ G : \mathbf{D}^b(\mathrm{Mod}_f A) \rightarrow \mathbf{D}(\mathrm{Mod} A)$$

is fully faithful. Suitable truncations ($\mathrm{smt}^{\geq d_0}$ and $\mathrm{smt}^{\leq d_1}$) show that the essential image of $F \circ G$ is $\mathbf{D}_f^b(\mathrm{Mod} A)$. \square

7.5. The Embedding of \mathbf{M} in $\mathbf{D}(\mathbf{M})$. Here again we only consider an abelian category \mathbf{M} .

For $M, N \in \mathbf{M}$ there is no difference between $\mathrm{Hom}_{\mathbf{M}}(M, N)$, $\mathrm{Hom}_{\mathbf{C}(\mathbf{M})}(M, N)$ and $\mathrm{Hom}_{\mathbf{K}(\mathbf{M})}(M, N)$. Thus the canonical functors $\mathbf{M} \rightarrow \mathbf{C}(\mathbf{M})$ and $\mathbf{M} \rightarrow \mathbf{K}(\mathbf{M})$ are fully faithful. The same is true for $\mathbf{D}(\mathbf{M})$, but this requires a proof.

Let $\mathbf{D}(\mathbf{M})^0$ be the full subcategory of $\mathbf{D}(\mathbf{M})$ consisting of complexes whose cohomology is concentrated in degree 0. This is an additive subcategory of $\mathbf{D}(\mathbf{M})$.

Proposition 7.5.1. *The canonical functor $\mathbf{M} \rightarrow \mathbf{D}(\mathbf{M})^0$ is an equivalence.*

Proof. Let's denote the canonical functor $\mathbf{M} \rightarrow \mathbf{D}(\mathbf{M})^0$ by F . Under the fully faithful embedding $\mathbf{M} \subseteq \mathbf{C}_{\mathrm{str}}(\mathbf{M})$, F is just the restriction of \tilde{Q} .

Since the functor $H^0 : \mathbf{D}(\mathbf{M}) \rightarrow \mathbf{M}$ satisfies $H^0 \circ F = \mathrm{Id}_{\mathbf{M}}$. This implies that F is faithful.

Next we prove that F is full. Take any objects $M, N \in \mathbf{M}$ and a morphism $q : M \rightarrow N$ in $\mathbf{D}(\mathbf{M})$. By Proposition 7.1.7 we know that $q = \tilde{Q}(a) \circ \tilde{Q}(s)^{-1}$ for some morphisms $a : L \rightarrow N$ and $s : L \rightarrow M$ in $\mathbf{C}_{\mathrm{str}}(\mathbf{M})$, with s a quasi-isomorphism. Let $L' := \mathrm{smt}^{\leq 0}(L)$, as in (7.3.6); so there is a quasi-isomorphism $u : L' \rightarrow L$ in $\mathbf{C}_{\mathrm{str}}(\mathbf{M})$. Writing $a' := a \circ u$ and $s' := s \circ u$, we see that s' is a quasi-isomorphism, and $q = \tilde{Q}(a') \circ \tilde{Q}(s')^{-1}$.

Next let $L'' := \mathrm{smt}^{\geq 0}(L')$, as in (7.3.8); so there is a surjective quasi-isomorphism $v : L' \rightarrow L''$ in $\mathbf{C}_{\mathrm{str}}(\mathbf{M})$. Because L'' is a complex concentrated in degree 0, we can view it as an object of \mathbf{M} . The morphisms a' and s' factor as $a' = a'' \circ v$ and $s' = s'' \circ v$, where $a'' : L'' \rightarrow N$ and $s'' : L'' \rightarrow M$ are morphisms in \mathbf{M} . But s'' is a quasi-isomorphism in $\mathbf{C}_{\mathrm{str}}(\mathbf{M})$, and so it is actually an isomorphism in \mathbf{M} . Therefore we have a morphism $a'' \circ (s'')^{-1}$ in \mathbf{M} , and

$$\tilde{Q}(a'' \circ (s'')^{-1}) = \tilde{Q}(a'') \circ \tilde{Q}(s'')^{-1} = \tilde{Q}(a') \circ \tilde{Q}(s')^{-1} = q.$$

Finally we have to prove that any $L \in \mathbf{D}(\mathbf{M})^0$ is isomorphic, in $\mathbf{D}(\mathbf{M})$, to a complex L'' that's concentrated in degree 0. But we already showed it in the previous paragraphs. \square

Proposition 7.5.2. *Let \mathbf{M} be an abelian category. Let*

$$0 \rightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \rightarrow 0$$

be a diagram in \mathbf{M} . The following conditions are equivalent:

- (i) *The diagram is an exact sequence.*
- (ii) *There is a distinguished triangle*

$$L \xrightarrow{\tilde{Q}(\phi)} M \xrightarrow{\tilde{Q}(\psi)} N \xrightarrow{\theta} T(L)$$

in $\mathbf{D}(\mathbf{M})$.

Exercise 7.5.3. Prove Proposition 7.5.2. (Hint: for the implication (i) \Rightarrow (ii) you can take $\theta = 0$.)

The last two propositions say that the abelian category \mathbf{M} can be recovered from the pretriangulated category $\mathbf{D}(\mathbf{M})$.

8. DERIVED FUNCTORS

As before, \mathbb{K} is a commutative base ring, that shall remain implicit. Let A be a central DG \mathbb{K} -ring, and \mathbf{M} a \mathbb{K} -linear abelian category. The category $\mathbf{C}(A, \mathbf{M})$ of DG A -modules in \mathbf{M} was introduced in Subsection 3.7. It is a DG category. The pretriangulated categories $\mathbf{K}(A, \mathbf{M})$ and $\mathbf{D}(A, \mathbf{M})$ were introduced in Subsections 5.4 and 7.1 respectively. There is a triangulated localization functor

$$Q : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{D}(A, \mathbf{M}).$$

Let (B, \mathbf{N}) be another pair of DG ring and abelian category. Suppose we are given a DG functor

$$F : \mathbf{C}(A, \mathbf{M}) \rightarrow \mathbf{C}(B, \mathbf{N}).$$

Then, according to Theorem 5.4.13, there is an induced triangulated functor

$$(\bar{F}, \bar{\tau}_F) : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{K}(B, \mathbf{N})$$

Most triangulated functors that we shall encounter arise this way. For convenience of notation, let us suppress mentioning the translation isomorphism $\bar{\tau}_F$, and let us write F instead of \bar{F} .

By postcomposing with the localization functor of $\mathbf{K}(B, \mathbf{N})$ we obtain a triangulated functor

$$(8.0.4) \quad Q \circ F : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{D}(B, \mathbf{N}).$$

Again we denote this triangulated functor by F .

Our goal in this section is to extend F to triangulated functors

$$RF, LF : \mathbf{D}(A, \mathbf{M}) \rightarrow \mathbf{D}(B, \mathbf{N}).$$

These are the right and left derived functors of F , respectively.

It will be easier to state matters more generally. Thus we shall mostly work in the setup below.

Setup 8.0.5. The following are given:

- (1) Pretriangulated categories \mathbf{K} and \mathbf{E} .
- (2) A triangulated functor $F : \mathbf{K} \rightarrow \mathbf{E}$.
- (3) A denominator set of cohomological origin $S \subseteq \mathbf{K}$ (see Definition 6.4.2).

Recall that the morphisms in S are called quasi-isomorphisms.

By Proposition 6.4.1 and Theorem 6.4.3, the localization \mathbf{K}_S exists, and it is a pretriangulated category. The triangulated localization functor is $Q : \mathbf{K} \rightarrow \mathbf{K}_S$.

This setup specializes to (8.0.4) when we take $\mathbf{K} = \mathbf{K}(A, \mathbf{M})$, $S = \mathbf{S}(A, \mathbf{M})$ and $\mathbf{E} = \mathbf{D}(B, \mathbf{N})$.

Remark 8.0.6. As far as we know, all previous textbooks only consider the special case of the derived functors

$$RF, LF : \mathbf{D}(\mathbf{M}) \rightarrow \mathbf{D}(\mathbf{N})$$

of a triangulated functor

$$F : \mathbf{K}(\mathbf{M}) \rightarrow \mathbf{K}(\mathbf{N}),$$

where \mathbf{M} and \mathbf{N} are abelian categories. The DG variant is not mentioned at all. However, the definitions and the main existence results, as stated in this section, are virtually the same.

Furthermore, previous textbooks avoid the 2-categorical notation, and that (in our opinion) is a cause for undue difficulties in the presentation.

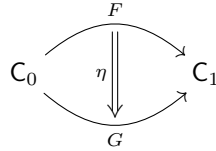
8.1. 2-Categorical Notation. In this section we are going to do a lot of work with morphisms of functors (i.e. natural transformations). The language and notation of ordinary category theory that we used so far is not adequate for this purpose. Therefore we will now introduce notation from the theory of 2-*categories*. (We will not give a definition of a 2-category here; but it is basically the data mentioned below, satisfying a few conditions, most of which will be mentioned below too.) In the subsequent sections we will revert to the usual (i.e. 1-categorical) language. For more details on 2-categories the reader can look at [Mac2] or [Ye10, Section 1].

Consider the set **Cat** of all categories. The set theoretical aspects are neglected, as explained in Subsection 1.1. (Briefly, the precise solution is this: **Cat** is the set of all **U**-categories; so **Cat** is a subset of a bigger Grothendieck universe, say **V**, and it is a **V**-category.)

The set **Cat** is the set of objects of a 2-category. This means that in **Cat** there are two kinds of morphisms: 1-*morphisms* between objects, and 2-*morphisms* between 1-morphisms. There are several kinds of compositions, and these have several properties. All this will be explained below.

Suppose C_0, C_1, \dots are categories, namely objects of **Cat**. The 1-morphisms between them are the functors. The notation is as usual: $F : C_0 \rightarrow C_1$ denotes a functor.

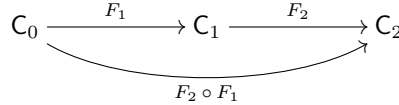
Suppose $F, G : C_0 \rightarrow C_1$ are functors (with the same source and target objects). The 2-morphisms from F to G are the morphisms of functors (i.e. the natural transformations), and the notation is $\eta : F \Rightarrow G$. The double arrow is the distinguishing notation for 2-morphisms. When specializing to an object $M \in C_0$ we revert to the single arrow notation, namely $\eta_M : F(M) \rightarrow G(M)$ is the corresponding morphism in C_1 . The diagram depicting this is



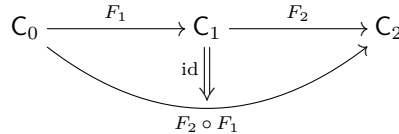
We shall refer to such a diagram as a 2-*diagram*.

Each object (category) C has its identity 1-morphism (functor) $\text{Id}_C : C \rightarrow C$. Each 1-morphism F has its identity 2-morphism (natural transformation) $\text{id}_F : F \Rightarrow F$.

Now we consider compositions. For functors there is nothing new: given functors $F_i : C_{i-1} \rightarrow C_i$, the composition, that we now call *horizontal composition*, is the functor $F_2 \circ F_1 : C_0 \rightarrow C_2$. The diagram is

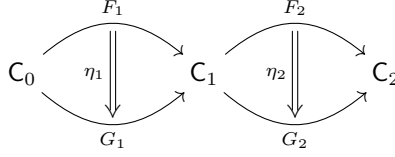


This can be viewed as a commutative 1-diagram, or as a shorthand for the 2-diagram



in which id is the identity 2-morphism of $F_2 \circ F_1$.

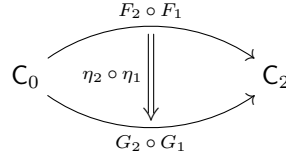
The complication begins with compositions of 2-morphisms. Suppose we are given 1-morphisms $F_i, G_i : \mathbf{C}_{i-1} \rightarrow \mathbf{C}_i$ and 2-morphisms $\eta_i : F_i \Rightarrow G_i$. In a diagram:



The *horizontal composition* is the morphism of functors

$$\eta_2 \circ \eta_1 : F_2 \circ F_1 \Rightarrow G_2 \circ G_1.$$

The diagram is

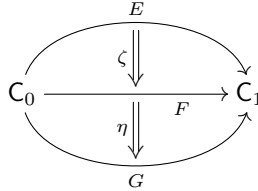


Exercise 8.1.1. For an object $M \in \mathbf{C}_0$, give an explicit formula for the morphism

$$(\eta_2 \circ \eta_1)_M : (F_2 \circ F_1)(M) \rightarrow (G_2 \circ G_1)(M)$$

in the category \mathbf{C}_2 .

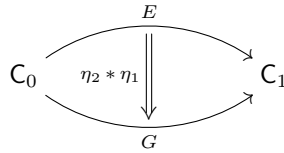
Suppose we are given 1-morphisms $E, F, G : \mathbf{C}_0 \rightarrow \mathbf{C}_1$, and 2-morphisms $\zeta : E \Rightarrow F$ and $\eta : F \Rightarrow G$. The diagram depicting this is



The *vertical composition* of ζ and η is the 2-morphism

$$\eta * \zeta : E \rightarrow G.$$

Notice the new symbol for this operation. The corresponding diagram is

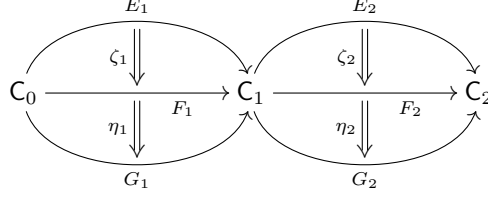


Exercise 8.1.2. For an object $M \in \mathbf{C}_0$, give an explicit formula for the morphism

$$(\eta * \zeta)_M : E(M) \rightarrow G(M)$$

in the category \mathbf{C}_1 .

Something intricate occurs in the situation shown in the next diagram.



It turns out that

$$(\eta_2 * \zeta_2) \circ (\eta_1 * \zeta_1) = (\eta_2 \circ \eta_1) * (\zeta_2 \circ \zeta_1)$$

as morphisms $E_2 \circ E_1 \Rightarrow G_2 \circ G_1$. This is called the *exchange property*.

Exercise 8.1.3. Prove the exchange property.

Just like general categories, we can talk about pretriangulated categories. There is the 2-category **PTrCat** of all pretriangulated categories (over \mathbb{K}). The objects here are the pretriangulated categories (\mathbf{K}, \mathbf{T}) ; the 1-morphisms are the triangulated functors (F, τ) ; and the 2-morphisms are the morphisms of triangulated functors η . This is what we are going to use.

8.2. Some Preliminaries on Triangulated Functors.

Proposition 8.2.1. *Let $(F, \tau) : \mathbf{K} \rightarrow \mathbf{L}$ be a triangulated functor between pretriangulated categories. Assume F is an equivalence (of abstract categories), with quasi-inverse $G : \mathbf{L} \rightarrow \mathbf{K}$, and with adjunction isomorphisms $\alpha : G \circ F \xrightarrow{\cong} \text{Id}_{\mathbf{K}}$ and $\beta : F \circ G \xrightarrow{\cong} \text{Id}_{\mathbf{L}}$.*

Then there is an isomorphism of functors

$$\nu : G \circ T_{\mathbf{L}} \xrightarrow{\cong} T_{\mathbf{K}} \circ G$$

such that $(G, \nu) : \mathbf{L} \rightarrow \mathbf{K}$ is a triangulated functor, and α and β are isomorphisms of triangulated functors.

Proof. It is well-known that G is additive (or in our case, \mathbb{K} -linear); but since the proof is so easy, we shall reproduce it. Take any pair of objects $M, N \in \mathbf{L}$. We have to prove that the bijection

$$G_{M,N} : \text{Hom}_{\mathbf{L}}(M, N) \rightarrow \text{Hom}_{\mathbf{K}}(G(M), G(N))$$

is linear. But

$$G_{M,N} = F_{G(M), G(N)}^{-1} \circ \text{Hom}_{\mathbf{L}}(\beta_M, \beta_N^{-1})$$

as bijections (of sets) between these modules. Since $\alpha_{M,N}^{-1}$ and $F_{G(M), G(N)}$ are \mathbb{K} -linear, then so is $G_{M,N}$.

We define the isomorphism of triangulated functors ν by the formula

$$\nu := (\alpha \circ \text{id}_{T_{\mathbf{K}} \circ G}) * (\text{id}_G \circ \tau \circ \text{id}_G)^{-1} * (\text{id}_{G \circ T_{\mathbf{L}}} \circ \beta)^{-1},$$

in terms of the 2-categorical notation. This gives rise to a commutative diagram of isomorphisms

$$\begin{array}{ccc} G \circ T_{\mathbf{L}} \circ F \circ G & \xleftarrow{\text{id} \circ \tau \circ \text{id}} & G \circ F \circ T_{\mathbf{K}} \circ G \\ \text{id} \circ \beta \downarrow & & \downarrow \alpha \circ \text{id} \\ G \circ T_{\mathbf{L}} & \xrightarrow{\nu} & T_{\mathbf{K}} \circ G \end{array}$$

of additive functors $\mathbf{L} \rightarrow \mathbf{K}$. So the pair (G, ν) is a T-additive functor.

The verification that (G, ν) preserves triangles (in the sense of Definition 5.3.1(1)) is done like the proof of the additivity of G , but now using axiom (TR1.a) from Definition 5.2.1. We leave this as an exercise. \square

Exercise 8.2.2. Finish the proof above (the last assertion).

8.3. Right Derived Functors.

Definition 8.3.1. Assume Setup 8.0.5. A *right derived functor* of F is a triangulated functor

$$RF : \mathbf{K}_S \rightarrow \mathbf{E},$$

together with a morphism

$$\eta : F \Rightarrow RF \circ Q$$

of triangulated functors $\mathbf{K} \rightarrow \mathbf{E}$. The pair (RF, η) must have this universal property:

- (\diamond) Given any pair (G, θ) , consisting of a triangulated functor $G : \mathbf{K}_S \rightarrow \mathbf{E}$ and a morphism of triangulated functors $\theta : F \Rightarrow G \circ Q$, there is a unique morphism of triangulated functors $\mu : RF \Rightarrow G$ such that $\theta = (\mu \circ \text{id}_Q) * \eta$.

Pictorially: there is a 2-diagram

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{F} & \mathbf{E} \\ \downarrow Q & \searrow RF & \uparrow \eta \\ \mathbf{K}_S & & \end{array}$$

For any other pair (G, θ) there is a unique morphism μ that sits in this 2-diagram:

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{F} & \mathbf{E} \\ \downarrow Q & \searrow \mu & \uparrow \theta \\ \mathbf{K}_S & \xrightarrow{G} & \end{array}$$

The 1-morphisms in this 2-diagram do not (necessarily) commute; but the diagram of 2-morphisms (with $*$ composition)

$$\begin{array}{ccc} F & & \\ \eta \downarrow & \searrow \theta & \\ RF \circ Q & \xrightarrow{\mu \circ \text{id}_Q} & G \circ Q \end{array}$$

is commutative.

Proposition 8.3.2. *If a right derived functor (RF, η) exists, then it is unique, up to a unique isomorphism. Namely, if (G, θ) is another right derived functor of F , then there is a unique isomorphism of triangulated functors $\mu : RF \xrightarrow{\cong} G$ such that $\theta = (\mu \circ \text{id}_Q) * \eta$.*

Proof. Despite the apparent complication of the situation, the usual argument for uniqueness of universals (here it is a universal 1-morphism) applies. It shows that the morphism μ from condition (\diamond) is an isomorphism. \square

Existence is much harder. Here is a sufficient condition. It is a rephrasing of [RD, Theorem I.5.1], and the proof is basically the same (but we give many more details).

Theorem 8.3.3. *Given Setup 8.0.5, assume there is a full pretriangulated subcategory $\mathcal{J} \subseteq \mathcal{K}$ with these two properties:*

- (a) *If $\phi : I \rightarrow I'$ is a quasi-isomorphism in \mathcal{J} , then $F(\phi) : F(I) \rightarrow F(I')$ is an isomorphism in \mathcal{E} .*
- (b) *Every object $M \in \mathcal{K}$ admits a quasi-isomorphism $\rho : M \rightarrow I$ to some object $I \in \mathcal{J}$.*

Then the right derived functor

$$(RF, \eta) : \mathcal{K}_{\mathcal{S}} \rightarrow \mathcal{E}$$

exists. Moreover, for any object $I \in \mathcal{J}$ the morphism

$$\eta_I : F(I) \rightarrow (RF \circ Q)(I)$$

in \mathcal{E} is an isomorphism.

Remark 8.3.4. A quasi-isomorphism $\rho : M \rightarrow I$ as in condition (b) is supposed to be viewed as a “generalized injective resolution” of M . See Example 8.3.22, where this is made concrete.

We use the letter \mathcal{J} for the category of “generalized injective complexes” because the letter \mathcal{I} , in this particular font, is too ambiguous.

The proof of the theorem follows some preparation. We will sometimes suppress the localization functors Q and Q' , for the sake of clarity. For instance, given a morphism $s \in \mathcal{S}$, we might say that s is invertible in $\mathcal{K}_{\mathcal{S}}$.

Definition 8.3.5. In the situation of Theorem 8.3.3, by a *system of right \mathcal{J} -resolutions* we mean a pair (I, ρ) , where $I : \text{Ob}(\mathcal{K}) \rightarrow \text{Ob}(\mathcal{J})$ is a function, and $\rho = \{\rho_M\}_{M \in \text{Ob}(\mathcal{K})}$ is a collection of quasi-isomorphisms $\rho_M : M \rightarrow I(M)$ in \mathcal{K} . Moreover, if $M \in \text{Ob}(\mathcal{J})$, then $I(M) = M$ and $\rho_M = \text{id}_M$.

Property (b) of Theorem 8.3.3 guarantees that a system of right \mathcal{J} -resolutions (I, ρ) exists.

Suppose we made a choice of a system of right \mathcal{J} -resolutions. Let us denote by $U : \mathcal{J} \rightarrow \mathcal{K}$ the inclusion functor, so $I \circ U$ is the identity on the set $\text{Ob}(\mathcal{J})$. Let us define $F' := F \circ U : \mathcal{J} \rightarrow \mathcal{E}$ and $\mathcal{S}' := \mathcal{J} \cap \mathcal{S}$. The localization functor of \mathcal{J} is denoted by $Q' : \mathcal{J} \rightarrow \mathcal{J}_{\mathcal{S}'}$. There is a triangulated functor $U_{\mathcal{S}'} : \mathcal{J}_{\mathcal{S}'} \rightarrow \mathcal{K}_{\mathcal{S}}$ extending U , and there is equality $Q \circ U = U_{\mathcal{S}'} \circ Q'$. These sit in a commutative diagram

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{U} & \mathcal{K} \\ Q' \downarrow & & \downarrow Q \\ \mathcal{J}_{\mathcal{S}'} & \xrightarrow{U_{\mathcal{S}'}} & \mathcal{K}_{\mathcal{S}} \end{array}$$

We know (from Theorem 6.4.3) that the functor F' extends uniquely to a triangulated functor $F'_{\mathcal{S}'} : \mathcal{J}_{\mathcal{S}'} \rightarrow \mathcal{E}$. Let $\eta' := \text{id}_{F'}$, which is a 2-morphism

$$(8.3.6) \quad \eta' : F' \Rightarrow F'_{\mathcal{S}'} \circ Q'.$$

The 2-diagram is:

$$(8.3.7) \quad \begin{array}{ccc} J & \xrightarrow{F'} & E \\ Q' \downarrow & \eta' \Downarrow & \nearrow F'_{S'} \\ J_{S'} & & \end{array}$$

Lemma 8.3.8. *The pair $(F'_{S'}, \eta')$ is a right derived functor of F' .*

Proof. We need to verify condition (\diamond) of Definition 8.3.1. Say a triangulated functor $G' : J_{S'} \rightarrow E$ is given. Because Q' is the identity on objects, the data of a morphism of triangulated functors $\mu' : F'_{S'} \Rightarrow G'$, namely a collection of morphisms $\mu'_I : F'(I) \rightarrow G'(I)$ in E for all $I \in J$, is the same data as a morphism of triangulated functors

$$(8.3.9) \quad \theta' := \mu' \circ \text{id}_{Q'} = (\mu' \circ \text{id}_{Q'}) * \eta' : F' \Rightarrow G' \circ Q'.$$

This implies that the function $\mu' \mapsto \theta'$ is injective. Here is the relevant 2-diagram:

$$(8.3.10) \quad \begin{array}{ccc} J & \xrightarrow{F'} & E \\ Q' \downarrow & \begin{array}{c} \eta' \Downarrow \\ F'_{S'} \nearrow \end{array} & \nearrow \\ J_{S'} & & \end{array} \quad \begin{array}{c} \mu' \Downarrow \\ G' \nearrow \end{array}$$

We have to prove that the function $\mu' \mapsto \theta'$ is surjective. This amounts to showing that for any morphism $q : I \rightarrow J$ in $J_{S'}$ there is equality

$$\theta'_J \circ F'_{S'}(q) = G'(q) \circ \theta'_I$$

of morphisms in E . Let us choose a right fraction presentation $q = a \circ s^{-1}$, with $a : K \rightarrow J$ in J and $s : K \rightarrow I$ in S' . Because $\theta' : F' \Rightarrow G' \circ Q'$ is a morphism of functors $J \rightarrow E$, the solid diagram below

$$\begin{array}{ccccc} & & F'_{S'}(q) & & \\ & \text{---} & \text{---} & \text{---} & \\ F'(I) & \xleftarrow{F'(s)} & F'(K) & \xrightarrow{F'(a)} & F'(J) \\ \theta'_I \downarrow & & \theta'_K \downarrow & & \theta'_J \downarrow \\ G'(I) & \xleftarrow{G'(s)} & G'(K) & \xrightarrow{G'(a)} & G'(J) \\ & \text{---} & \text{---} & \text{---} & \\ & & G'(q) & & \end{array}$$

is commutative. But then, since $F'(s)$ and $G'(s)$ are invertible in E , the whole diagram is commutative. \square

Lemma 8.3.11. *The functor $U_{S'} : J_{S'} \rightarrow K_S$ is an equivalence of pretriangulated categories.*

Proof. By the proof of Proposition 7.2.5, with condition (1), together with Proposition 8.2.1. \square

Lemma 8.3.12. *Suppose a system of right \mathbf{J} -resolutions (I, ρ) has been chosen. Then the function I extends uniquely to a triangulated functor $I : \mathbf{K}_S \rightarrow \mathbf{J}_{S'}$, such that $\text{Id}_{\mathbf{J}_{S'}} = I \circ U_{S'}$, and $\rho : \text{Id}_{\mathbf{K}_S} \Rightarrow U_{S'} \circ I$ is an isomorphism of triangulated functors.*

In other words, the triangulated functor I is a quasi-inverse of $U_{S'}$. The relevant 2-diagram is this:

$$\begin{array}{ccccc}
 \mathbf{J} & \xrightarrow{Q'} & \mathbf{J}_{S'} & \xrightarrow{\text{Id}} & \mathbf{J}_{S'} \\
 \downarrow U & & \uparrow I & \searrow U_{S'} & \uparrow I \\
 \mathbf{K} & \xrightarrow{Q} & \mathbf{K}_S & \xrightarrow{\text{Id}} & \mathbf{K}_S \\
 & & \uparrow \rho & &
 \end{array}$$

Proof. By Lemma 8.3.11 the functor $U_{S'}$ is an equivalence. Take any pair of objects $M, N \in \mathbf{K}$. There is a bijection

$$U_{S'} : \text{Hom}_{\mathbf{J}_{S'}}(I(M), I(N)) \rightarrow \text{Hom}_{\mathbf{K}_S}(I(M), I(N)),$$

and another bijection

$$\text{Hom}(\rho_M^{-1}, \rho_N) : \text{Hom}_{\mathbf{K}_S}(M, N) \rightarrow \text{Hom}_{\mathbf{K}_S}(I(M), I(N)).$$

These bijections say that to any morphism $\psi : M \rightarrow N$ in \mathbf{K}_S there corresponds a unique morphism $I(\psi) : I(M) \rightarrow I(N)$ in $\mathbf{J}_{S'}$, such that

$$U_{S'}(I(\psi)) \circ \rho_M = \rho_N \circ \psi.$$

An easy calculation shows that $I : \mathbf{K}_S \rightarrow \mathbf{J}_{S'}$ is a functor. Moreover, there is equality of functors $I \circ U_{S'} = \text{Id}_{\mathbf{J}_{S'}}$, and an isomorphism of functors $\rho : \text{Id}_{\mathbf{K}_S} \xrightarrow{\cong} U_{S'} \circ I$. This says that I is a quasi-inverse of $U_{S'}$. Therefore, by Proposition 8.2.1, I is a triangulated functor, and ρ is an isomorphism of triangulated functors. \square

Lemma 8.3.13. *Under the assumptions of the theorem, let $G : \mathbf{K}_S \rightarrow \mathbf{E}$ be triangulated functor, and define $G' := G \circ U_{S'}$. Suppose $\eta' : F' \Rightarrow G' \circ Q'$ is a morphism of triangulated functors $\mathbf{J} \rightarrow \mathbf{E}$. Then there is a unique morphism $\eta : F \Rightarrow G \circ Q$ of triangulated functors $\mathbf{K} \rightarrow \mathbf{E}$ that extends η' , namely such that $\eta \circ \text{id}_U = \eta'$.*

Here are the corresponding 2-diagrams:

$$\begin{array}{ccc}
 \mathbf{J} & \xrightarrow{F'} & \mathbf{E} \\
 \downarrow Q' & \searrow \eta' & \uparrow G' \\
 \mathbf{J}_{S'} & &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \mathbf{J} & \xrightarrow{U} & \mathbf{K} & \xrightarrow{F} & \mathbf{E} \\
 \downarrow Q' & & \downarrow Q & \searrow \eta & \uparrow G \\
 \mathbf{J}_{S'} & \xrightarrow{U_{S'}} & \mathbf{K}_S & &
 \end{array}$$

Here is another way to state the lemma. Let us denote by $\text{Hom}_{\mathbf{PTrCat}}^2(-, -)$ the set of 2-morphisms (morphisms of triangulated functors). Then the operation $\eta \mapsto \eta \circ \text{id}_U$ is a function

$$- \circ \text{id}_U : \text{Hom}_{\mathbf{PTrCat}}^2(F, G \circ Q) \rightarrow \text{Hom}_{\mathbf{PTrCat}}^2(F', G' \circ Q'),$$

and the lemma asserts that this is a bijection.

Proof. Choose a system of right J-resolutions (I, ρ) . For any object $M \in \mathbf{K}$ the morphism ρ_M is invertible in \mathbf{K}_S . Hence the morphism

$$G(\rho_M) : G(M) \rightarrow G(I(M))$$

is invertible in \mathbf{E} . We are given the morphism

$$\eta'_{I(M)} : F'(I(M)) \rightarrow G'(I(M))$$

in \mathbf{E} . Recall that $F'(I(M)) = F(I(M))$ and $G'(I(M)) = G(I(M))$. Let us define

$$(8.3.14) \quad \eta_M := G(\rho_M)^{-1} \circ \eta'_{I(M)} \circ F(\rho_M),$$

which is a morphism $F(M) \rightarrow G(M)$ in \mathbf{E} . We get a commutative diagram

$$(8.3.15) \quad \begin{array}{ccc} F(M) & \xrightarrow{\eta_M} & G(M) \\ F(\rho_M) \downarrow & & \downarrow G(\rho_M) \\ F'(I(M)) & \xrightarrow{\eta'_{I(M)}} & G'(I(M)) \end{array}$$

in \mathbf{E} .

It is now routine to check that η is a morphism of triangulated functors $F \Rightarrow G \circ Q$. By construction η extends η' . The uniqueness of η follows from the fact that the diagram (8.3.15) must commute, and thus formula (8.3.14) must hold. \square

Proof of Theorem 8.3.3.

Step 1. We choose a system of right J-resolutions (I, ρ) . For any object $M \in \mathbf{K}$ we define the object

$$(8.3.16) \quad \mathbf{R}F(M) := F(I(M)) \in \mathbf{E}$$

and the morphism

$$(8.3.17) \quad \eta_M := F(\rho_M) : F(M) \rightarrow \mathbf{R}F(M)$$

in \mathbf{E} . We still did not say what $\mathbf{R}F$ does to morphisms.

Step 2. For any object $M \in \mathbf{K}$ we have, by construction, $\mathbf{R}F(M) = F'(I(M))$. This means that $\mathbf{R}F = F'_{S'} \circ I$ on objects. The definition

$$(8.3.18) \quad \mathbf{R}F := F'_{S'} \circ I : \mathbf{K}_S \rightarrow \mathbf{E}.$$

upgrades $\mathbf{R}F$ to a triangulated functor. And there is a commutative diagram of triangulated functors

$$\begin{array}{ccccc} & & F' & & \\ & \curvearrowright & & \curvearrowright & \\ \mathbf{J} & \xrightarrow{Q'} & \mathbf{J}_{S'} & \xrightarrow{F'_{S'}} & \mathbf{E} \\ \downarrow U & & \uparrow I & \nearrow \mathbf{R}F & \\ \mathbf{K} & \xrightarrow{Q} & \mathbf{K}_S & & \end{array}$$

Step 3. Recall that we already defined $\eta_M = F(\rho_M)$. In this step we prove that η is a morphism of triangulated functors $\eta : F \rightarrow \mathbf{R}F \circ Q$.

According to Lemma 8.3.13, the morphism of triangulated functors $\eta' : F' \Rightarrow F'_{S'} \circ Q'$ from (8.3.6) extends uniquely to a morphism of triangulated functors $\tilde{\eta} : F \Rightarrow RF \circ Q$. The 2-diagram is

$$\begin{array}{ccc} K & \xrightarrow{F} & E \\ \downarrow Q & \searrow \tilde{\eta} & \uparrow RF \\ K_S & & \end{array}$$

We know that $\eta'_{I(M)} = \text{id}_{F(I(M))}$ and $RF = F'_{S'} \circ I$. By construction of the functor I we have $I(\rho_M) = \text{id}_{I(M)}$ in $J_{S'}$. Plugging this and $G = RF$ into formula (8.3.14) we obtain

$$\begin{aligned} \tilde{\eta}_M &= (F'_{S'}(I(\rho_M)))^{-1} \circ \eta'_{I(M)} \circ F(\rho_M) \\ &= (\text{id}_{F(I(M))})^{-1} \circ \text{id}_{F(I(M))} \circ F(\rho_M) = F(\rho_M). \end{aligned}$$

So the morphism $\tilde{\eta}_M$ coincides with η_M . As M varies we get $\tilde{\eta} = \eta$.

Step 4. It remains to verify condition (\diamond) of Definition 8.3.1. Say a pair (G, θ) is given. Define $G' := G \circ U_{S'}$ and $\theta' := \theta \circ \text{id}_U$. In Lemma 8.3.8 we proved that $(F'_{S'}, \eta')$ is the right derived functor of F' . Therefore there is a unique morphism $\mu' : F'_{S'} \Rightarrow G'$ of triangulated functors $J_{S'} \rightarrow E$ such that $\mu' \circ \text{id}_{Q'} = \theta'$. In terms of vertical composition, and using the equality $\eta' = \text{id}_{F'}$, this is

$$(8.3.19) \quad (\mu' \circ \text{id}_{Q'}) * \eta' = \theta'.$$

In a 2-diagram:

$$\begin{array}{ccc} J & \xrightarrow{F'} & E \\ \downarrow Q' & \searrow \eta' & \uparrow \theta' \\ J_{S'} & \xrightarrow{F'_{S'}} & E \end{array}$$

(Note: The diagram above is a simplified representation of the 2-diagram in the image, which includes a curved arrow G' from $J_{S'}$ to E and a morphism μ' from $F'_{S'}$ to G' .)

Recall that $F'_{S'} = RF \circ U_{S'}$. The functor $U_{S'}$ is an equivalence. Hence (like Lemma 8.3.13 but much easier) there is a unique morphism $\mu : RF \rightarrow G$ such that $\mu \circ \text{id}_{U_{S'}} = \mu'$. We get this 2-diagram:

$$\begin{array}{ccccc} J & \xrightarrow{U} & K & \xrightarrow{F} & E \\ \downarrow Q' & & \downarrow Q & \searrow \eta & \uparrow \theta \\ J_{S'} & \xrightarrow{U_{S'}} & K_S & \xrightarrow{RF} & E \end{array}$$

(Note: The diagram above is a simplified representation of the 2-diagram in the image, which includes a curved arrow G from K_S to E and a morphism μ from RF to G .)

We know that

$$\text{id}_Q \circ \text{id}_U = \text{id}_{U_{S'}} \circ \text{id}_{Q'}.$$

Hence

$$(\mu \circ \text{id}_Q \circ \text{id}_U) * (\eta \circ \text{id}_U) = (\mu \circ \text{id}_{U_{S'}} \circ \text{id}_{Q'}) * \eta' = (\mu' \circ \text{id}_{Q'}) * \eta'$$

(this is the exchange condition). Taking this with formula (8.3.19), and using the exchange condition once more, we deduce that

$$((\mu \circ \text{id}_Q) * \eta) \circ \text{id}_U = \theta'.$$

The uniqueness in Lemma 8.3.13 now implies that

$$(8.3.20) \quad (\mu \circ \text{id}_Q) * \eta = \theta.$$

Finally we have to establish the uniqueness of μ . Suppose $\tilde{\mu}$ is another morphism $RF \Rightarrow G$ satisfying (8.3.20). Then $\tilde{\mu}' := \tilde{\mu} \circ \text{id}_{U_{S'}}$ satisfies (8.3.19). But then, by the uniqueness of μ' , we have $\tilde{\mu}' = \mu'$. Therefore (because $U_{S'}$ is an equivalence) we see that $\tilde{\mu} = \mu$. \square

Definition 8.3.21. The construction of the right derived functor (RF, η) in the proof of the theorem above, and specifically formulas (8.3.16) and (8.3.17), is called a *presentation of (RF, η) by the system of right J-resolutions (I, ρ)* .

Of course any other right derived functor of F (perhaps presented by another system of right J-resolutions) is uniquely isomorphic to (RF, η) . This is according to Proposition 8.3.2.

In Section 9 we shall give several existence results for the right derived functor

$$(RF, \eta) : \mathbf{D}^*(A, M) \rightarrow \mathbf{E}$$

of a triangulated functor

$$F : \mathbf{K}^*(A, M) \rightarrow \mathbf{E},$$

under various assumptions on F , A , M and \star . These existence results will be based on Theorem 8.3.3: we will prove existence of suitable resolving subcategories $\mathbf{J} \subseteq \mathbf{K}^*(A, M)$. The example below is one such case.

Example 8.3.22. Suppose we start from an additive functor $F : M \rightarrow N$. We know how to extend it to a DG functor $F : \mathbf{C}^+(M) \rightarrow \mathbf{C}^+(N)$, and then to a triangulated functor $F : \mathbf{K}^+(M) \rightarrow \mathbf{K}^+(N)$. By composing with Q we get a triangulated functor $Q \circ F : \mathbf{K}^+(M) \rightarrow \mathbf{D}^+(N)$, that we also denote by F for simplicity.

Assume that the abelian category M has enough injectives (this means that any object $M \in M$ admits an injective resolution). Define \mathbf{J} to be the full subcategory of $\mathbf{K} := \mathbf{K}^+(M)$ on the bounded below complexes of injective objects; and let $\mathbf{E} := \mathbf{D}^+(N)$. We will prove later that properties (a) and (b) of Theorem 8.3.3 hold in this situation. Therefore we have a right derived functor

$$RF : \mathbf{D}^+(M) \rightarrow \mathbf{D}^+(N).$$

In case the functor F is left exact, it has the classical right derived functors $R^q F : M \rightarrow N$, $q \geq 0$. Formula (8.3.16) shows that for any $M \in M$ there is equality $R^q F(M) = H^q(RF(M))$ as objects of N . We will prove that more is true:

$$R^q F = H^q \circ RF$$

as functors $M \rightarrow N$.

In the situation of Theorem 8.3.3, let \mathbf{K}^\dagger be a full pretriangulated subcategory of \mathbf{K} . Define $\mathbf{S}^\dagger := \mathbf{K}^\dagger \cap \mathbf{S}$ and $\mathbf{J}^\dagger := \mathbf{K}^\dagger \cap \mathbf{J}$. Denote by $V : \mathbf{K}^\dagger \rightarrow \mathbf{K}$ the inclusion functor, and by $V_{\mathbf{S}^\dagger} : \mathbf{K}_{\mathbf{S}^\dagger}^\dagger \rightarrow \mathbf{K}_{\mathbf{S}}$ its localization. Warning: the functor $V_{\mathbf{S}^\dagger}$ is not necessarily fully faithful; cf. Proposition 7.2.5.

Proposition 8.3.23. *Assume that every $M \in \mathbf{K}^\dagger$ admits a quasi-isomorphism $M \rightarrow I$ where $I \in \mathbf{J}^\dagger$. Then the pair*

$$(RF \circ V_{S^\dagger}, \eta \circ \text{id}_V)$$

is a right derived functor of $F \circ V : \mathbf{K}^\dagger \rightarrow \mathbf{E}$.

Loosely speaking, the proposition says that

$$R(F \circ V) = RF \circ V_{S^\dagger}.$$

The proof is an exercise.

Exercise 8.3.24. Prove the last proposition. (Hint: Start by choosing a system of right \mathbf{J}^\dagger -resolutions of \mathbf{K}^\dagger . Then extend it to a system of right \mathbf{J} -resolutions of \mathbf{K} . Now follow the proof of the theorem.)

8.4. Left Derived Functors. Left derived functors behave just like right derived functors, except for a change of sides in the target category. Because of this our treatment will be brief: we will state the definitions and the main results, but won't give proofs, beyond a hint here and there.

Definition 8.4.1. Assume Setup 8.0.5. A *left derived functor* of F is a triangulated functor

$$LF : \mathbf{K}_S \rightarrow \mathbf{E},$$

together with a morphism

$$\eta : LF \circ Q \Rightarrow F$$

of triangulated functors $\mathbf{K} \rightarrow \mathbf{E}$. The pair (LF, η) must have this universal property:

- (\diamond) Given any pair (G, θ) , consisting of a triangulated functor $G : \mathbf{K}_S \rightarrow \mathbf{E}$ and a morphism of triangulated functors $\theta : G \circ Q \Rightarrow F$, there is a unique morphism of triangulated functors $\mu : G \Rightarrow LF$ such that $\theta = \eta * (\mu \circ \text{id}_Q)$.

Pictorially: there is a 2-diagram

For any other pair (G, θ) there is a unique morphism μ that sits in this 2-diagram:

The 1-morphisms in this 2-diagram do not (necessarily) commute; but the diagram of 2-morphisms (with $*$ composition)

$$\begin{array}{ccc}
 & F & \\
 \eta \uparrow & \swarrow \theta & \\
 LF \circ Q & \xleftarrow{\mu \circ \text{id}_Q} & G \circ Q
 \end{array}$$

is commutative.

Proposition 8.4.2. *If a left derived functor (LF, η) exists, then it is unique, up to a unique isomorphism. Namely, if (G, θ) is another left derived functor of F , then there is a unique isomorphism of triangulated functors $\mu : G \xrightarrow{\cong} LF$ such that $\theta = \eta * (\mu \circ \text{id}_Q)$.*

The proof is the same as that of Proposition 8.3.2, with direction of arrows in \mathbf{E} reversed.

Theorem 8.4.3. *Given Setup 8.0.5, assume there is a full pretriangulated subcategory $\mathbf{P} \subseteq \mathbf{K}$ with these two properties:*

- (a) *If $\phi : P \rightarrow P'$ is a quasi-isomorphism in \mathbf{P} , then $F(\phi) : F(P) \rightarrow F(P')$ is an isomorphism in \mathbf{E} .*
- (b) *Every object $M \in \mathbf{K}$ admits a quasi-isomorphism $\rho : P \rightarrow M$ from some object $P \in \mathbf{P}$.*

Then the right derived functor

$$(LF, \eta) : \mathbf{K}_S \rightarrow \mathbf{E}$$

exists. Moreover, for any object $P \in \mathbf{P}$ the morphism

$$\eta_P : (LF \circ Q)(P) \rightarrow F(P)$$

in \mathbf{E} is an isomorphism.

The category \mathbf{P} is a “generalized category of projectives”.

The proof is the same as that of Theorem 8.3.3, with direction of arrows in \mathbf{E} reversed.

Definition 8.4.4. In the situation of Theorem 8.4.3, by a *system of left \mathbf{P} -resolutions* we mean a pair (P, ρ) , where $P : \text{Ob}(\mathbf{K}) \rightarrow \text{Ob}(\mathbf{P})$ is a function, and $\rho = \{\rho_M\}_{M \in \text{Ob}(\mathbf{K})}$ is a collection of quasi-isomorphisms $\rho_M : P(M) \rightarrow M$ in \mathbf{K} . Moreover, if $M \in \text{Ob}(\mathbf{P})$, then $P(M) = M$ and $\rho_M = \text{id}_M$.

Property (b) of Theorem 8.4.3 guarantees that a system of left \mathbf{P} -resolutions (P, ρ) exists.

Definition 8.4.5. The construction of the left derived functor (LF, η) , when proving Theorem 8.4.3 along the lines of Theorem 8.3.3, and specifically the formulas

$$(8.4.6) \quad LF(M) := F(P(M))$$

and

$$\eta_M := F(\rho_M) : LF(M) \rightarrow F(M),$$

is called a *presentation of (LF, η) by the system of left \mathbf{P} -resolutions (P, ρ) .*

In Section 9 we shall give several existence results for the left derived functor

$$(LF, \eta) : \mathbf{D}^*(A, M) \rightarrow E$$

of a triangulated functor

$$F : \mathbf{K}^*(A, M) \rightarrow E,$$

under various assumptions on F , A , M and \star . These existence results will be based on Theorem 8.4.3: we will prove existence of suitable resolving subcategories $P \subseteq \mathbf{K}^*(A, M)$. The example below is one such case.

Example 8.4.7. Suppose we start from an additive functor $F : M \rightarrow N$. We know how to extend it to a DG functor $F : \mathbf{C}^-(M) \rightarrow \mathbf{C}^-(N)$, and then to a triangulated functor $F : \mathbf{K}^-(M) \rightarrow \mathbf{K}^-(N)$. By composing with Q we get a triangulated functor $Q \circ F : \mathbf{K}^-(M) \rightarrow \mathbf{D}^-(N)$, that we also denote by F for simplicity.

Assume that the abelian category M has enough projectives (this means that any object $M \in M$ admits a projective resolution). Define P to be the full subcategory of $K := \mathbf{K}^-(M)$ on the bounded above complexes of projective objects; and let $E := \mathbf{D}^-(N)$. We will prove later that properties (a) and (b) of Theorem 8.4.3 hold in this situation. Therefore we have a left derived functor

$$LF : \mathbf{D}^-(M) \rightarrow \mathbf{D}^-(N).$$

In case the functor F is right exact, it has the classical left derived functors $L_q F : M \rightarrow N$, $q \geq 0$. Formula (8.4.6) shows that for any $M \in M$ there is equality $L_q F(M) = H^{-q}(LF(M))$ as objects of N . We will prove that more is true:

$$L_q F = H^{-q} \circ LF$$

as functors $M \rightarrow N$.

In the situation of Theorem 8.4.3, let K^\dagger be a full pretriangulated subcategory of K . Define $S^\dagger := K^\dagger \cap S$ and $P^\dagger := K^\dagger \cap P$. Denote by $V : K^\dagger \rightarrow K$ the inclusion functor, and by $V_{S^\dagger} : K_{S^\dagger}^\dagger \rightarrow K_S$ its localization. Warning: the functor V_{S^\dagger} is not necessarily fully faithful; cf. Proposition 7.2.5.

Proposition 8.4.8. *Assume that every $M \in K^\dagger$ admits a quasi-isomorphism $P \rightarrow M$ where $P \in P^\dagger$. Then the pair*

$$(LF \circ V_{S^\dagger}, \eta \circ \text{id}_V)$$

is a left derived functor of $F \circ V : K^\dagger \rightarrow E$.

The proof is just like that of Proposition 8.3.23 (which was an exercise...).

9. RESOLUTIONS OF DG MODULES

In this section we are back to the more concrete setting: A is a DG ring, and \mathbf{M} is an abelian category (both over a base ring \mathbb{K}). We will define *K-projective* and *K-injective* DG modules in $\mathbf{K}(A, \mathbf{M})$. These DG modules form full pretriangulated subcategories of $\mathbf{K}(A, \mathbf{M})$, and are concrete versions of the abstract categories \mathbf{J} and \mathbf{P} , that played important roles in Subsections 8.3 and 8.4 respectively. For $\mathbf{K}(A)$ we also define *K-flat DG modules*.

9.1. K-Injective DG Modules. For any i we have an additive functor

$$H^i : \mathbf{C}_{\text{str}}(A, \mathbf{M}) \rightarrow \mathbf{M}.$$

There is equality $H^i = H^0 \circ T^i$. The functors H^i pass to the homotopy category, and

$$H^0 : \mathbf{K}(A, \mathbf{M}) \rightarrow \mathbf{M}$$

is a cohomological functor in the sense of Definition 5.3.2.

Definition 9.1.1. A DG module $N \in \mathbf{C}(A, \mathbf{M})$ is called *acyclic* if $H^i(N) = 0$ for all i .

Definition 9.1.2. A DG module $I \in \mathbf{C}(A, \mathbf{M})$ is called *K-injective* if for every acyclic DG module $N \in \mathbf{C}(A, \mathbf{M})$, the DG \mathbb{K} -module $\text{Hom}_{A, \mathbf{M}}(N, I)$ is acyclic.

The definition above characterizes K-injectives as objects of $\mathbf{C}(A, \mathbf{M})$. The next proposition shows that being K-injective is intrinsic to the pretriangulated category $\mathbf{K}(A, \mathbf{M})$, with the cohomological functor H^0 (that tells us which are the acyclic objects).

Proposition 9.1.3. A DG module $I \in \mathbf{K}(A, \mathbf{M})$ is K-injective if and only if $\text{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, I) = 0$ for every acyclic DG module $N \in \mathbf{K}(A, \mathbf{M})$.

Proof. This is because for any integer p we have

$$H^p(\text{Hom}_{A, \mathbf{M}}(N, I)) \cong H^0(\text{Hom}_{A, \mathbf{M}}(T^{-p}(N), I)) \cong \text{Hom}_{\mathbf{K}(A, \mathbf{M})}(T^{-p}(N), I),$$

and N is acyclic iff $T^{-p}(N)$ is acyclic. \square

The concept of K-injective complex (i.e. a K-injective object of $\mathbf{K}(\mathbf{M})$) was introduced by Spaltenstein [Sp] in 1988. At about the same time other authors (Keller [Ke], Bockstedt-Neeman [BoNe], Bernstein-Lunts [BeLu], ...) discovered this concept independently, with other names (such as *homotopically injective complex*). The texts [BeLu] and [Ke] already talk about DG modules over DG rings.

Remark 9.1.4. When the smart truncation functors exist (e.g. when A is a nonpositive DG ring), it is enough to check for K-injectivity of a DG module $I \in \mathbf{K}^*(A, \mathbf{M})$ against acyclic DG modules $N \in \mathbf{K}^*(A, \mathbf{M})$. Cf. Definition 7.3.11 and Exercise 7.3.12.

Definition 9.1.5. Let $M \in \mathbf{K}(A, \mathbf{M})$. A *K-injective resolution* of M is a quasi-isomorphism $\rho : M \rightarrow I$ in $\mathbf{K}(A, \mathbf{M})$, where I is a K-injective DG module.

Remark 9.1.6. In some other texts (and in our Section 10) “resolution” refers to a quasi-isomorphism $\rho : M \rightarrow I$ in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. It usually makes no difference which meaning is used (as long as we know what we are talking about).

In the next section we will prove existence of K -injectives in several contexts. Here is an easy one.

Exercise 9.1.7. Let $I \in \mathbf{K}(\mathbf{M})$ be a complex of injective objects of \mathbf{M} , with zero differential. Prove that I is K -injective.

Definition 9.1.8. Let K be a full subcategory of $\mathbf{K}(A, \mathbf{M})$. The full subcategory of K on the K -injective DG modules in it is denoted by K_{inj} . In other words,

$$K_{\text{inj}} = \mathbf{K}(A, \mathbf{M})_{\text{inj}} \cap K.$$

Warning: the property of being K -injective is in general not intrinsic to the subcategory K . Cf. Remark 9.1.4.

Proposition 9.1.9. *If K is a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$, then K_{inj} is a full pretriangulated subcategory of K .*

Proof. It suffices to prove that $\mathbf{K}(A, \mathbf{M})_{\text{inj}}$ is a pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. It is easy to see that $\mathbf{K}(A, \mathbf{M})_{\text{inj}}$ is closed under translations. Suppose

$$I \rightarrow J \rightarrow K \rightarrow T(I)$$

is a distinguished triangle in $\mathbf{K}(A, \mathbf{M})$, with I, J being K -injective DG modules. We have to show that K is also K -injective. Take any acyclic DG module $N \in \mathbf{K}(A, \mathbf{M})$. There is an exact sequence

$$\text{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, J) \rightarrow \text{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, K) \rightarrow \text{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, T(I))$$

in $\text{Mod } \mathbb{K}$. Because J and $T(I)$ are K -injectives, Proposition 9.1.3 says that

$$\text{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, J) = 0 = \text{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, T(I)).$$

Therefore $\text{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, K) = 0$. But N is an arbitrary acyclic DG module, so K is K -injective. \square

Example 9.1.10. Let \star be some boundedness condition (namely b , $+$, $-$ or nothing). We know that $\mathbf{K}^\star(A, \mathbf{M})$ is a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. Hence $\mathbf{K}^\star(A, \mathbf{M})_{\text{inj}}$ is a pretriangulated subcategory too.

Definition 9.1.11. Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. We say that K *has enough K -injectives* if any DG module $M \in K$ admits a K -injective resolution inside K . I.e. there is a quasi-isomorphism $\rho : M \rightarrow I$ where $I \in K_{\text{inj}}$.

Here is the crucial fact regarding K -injectives.

Lemma 9.1.12. *Let K be a full subcategory of $\mathbf{K}(A, \mathbf{M})$. Let $s : I \rightarrow M$ be a quasi-isomorphism in K , and assume I is K -injective. Then s has a left inverse, namely there is a morphism $t : M \rightarrow I$ in K such that $t \circ s = \text{id}_I$.*

Proof. Since K is a full subcategory of $\mathbf{K}(A, \mathbf{M})$, we can assume that $K = \mathbf{K}(A, \mathbf{M})$. Consider a distinguished triangle

$$I \xrightarrow{s} M \rightarrow N \rightarrow T(I)$$

in $\mathbf{K}(A, \mathbf{M})$ that's built on s . The long exact cohomology sequence tells us that N is an acyclic DG module. So

$$\text{Hom}_{\mathbf{K}(A, \mathbf{M})}(T^p(N), I) = 0$$

for all p . The exact sequence

$$\begin{aligned} \mathrm{Hom}_{\mathbf{K}(A, \mathbf{M})}(N, I) &\rightarrow \mathrm{Hom}_{\mathbf{K}(A, \mathbf{M})}(M, I) \\ &\rightarrow \mathrm{Hom}_{\mathbf{K}(A, \mathbf{M})}(I, I) \rightarrow \mathrm{Hom}_{\mathbf{K}(A, \mathbf{M})}(T^{-1}(N), I) \end{aligned}$$

shows that $\phi \mapsto \phi \circ s$ is a bijection

$$\mathrm{Hom}_{\mathbf{K}(A, \mathbf{M})}(M, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(A, \mathbf{M})}(I, I).$$

We take $t : M \rightarrow I$ to be the unique morphism in $\mathbf{K}(A, \mathbf{M})$ such that $t \circ s = \mathrm{id}_I$. \square

Theorem 9.1.13. *Let A be a DG ring, let \mathbf{M} be an abelian category, and let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. Denote by \mathbf{S} the set of quasi-isomorphisms in \mathbf{K} . Then the localization functor*

$$Q : \mathbf{K}_{\mathrm{inj}} \rightarrow \mathbf{K}_{\mathbf{S}}$$

is fully faithful.

Proof. Consider any pair of objects $I, J \in \mathbf{K}_{\mathrm{inj}}$. We must prove that the \mathbb{K} -module homomorphism

$$(9.1.14) \quad Q : \mathrm{Hom}_{\mathbf{K}}(I, J) \rightarrow \mathrm{Hom}_{\mathbf{K}_{\mathbf{S}}}(I, J)$$

is bijective.

Suppose $q : I \rightarrow J$ is a morphism in $\mathbf{K}_{\mathbf{S}}$. Let us present q as a left fraction: $q = Q(s)^{-1} \circ Q(a)$, where $a : I \rightarrow N$ and $s : J \rightarrow N$ are morphisms in \mathbf{K} , and s is a quasi-isomorphism. By Lemma 9.1.12 s has a left inverse t . We get a morphism $t \circ a : I \rightarrow J$ in \mathbf{K} , and an easy calculation shows that $Q(t \circ a) = q$ in $\mathbf{K}_{\mathbf{S}}$. This proves surjectivity of (9.1.14).

Now let's prove injectivity of (9.1.14). If $a : I \rightarrow J$ is a morphism in \mathbf{K} such that $Q(a) = 0$, then by axiom (LO4) of Ore localization (the left version of axiom (RO4) in Definition 6.2.1), there is a quasi-isomorphism $s : J \rightarrow L$ in \mathbf{K} such that $s \circ a = 0$ in \mathbf{K} . Let t be the left inverse of s . Then $a = t \circ s \circ a = 0$ in \mathbf{K} . \square

Corollary 9.1.15. *Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. If \mathbf{K} has enough K -injectives, then the localization functor*

$$Q : \mathbf{K}_{\mathrm{inj}} \rightarrow \mathbf{K}_{\mathbf{S}}$$

is an equivalence.

Proof. By the theorem the functor Q is fully faithful. The extra condition guarantees that Q is essentially surjective on objects. \square

Corollary 9.1.16. *Let \star be any boundedness condition. If $\mathbf{K}^{\star}(A, \mathbf{M})$ has enough K -injectives, then the triangulated functor*

$$Q : \mathbf{K}^{\star}(A, \mathbf{M})_{\mathrm{inj}} \rightarrow \mathbf{D}^{\star}(A, \mathbf{M})$$

is an equivalence.

Proof. Since $\mathbf{K}^{\star}(A, \mathbf{M})$ is a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$, this is a special case of the previous corollary. \square

Remark 9.1.17. This result is of tremendous importance, both theoretically and practically. In the theory, it shows that the localized category $\mathbf{D}^{\star}(A, \mathbf{M})$, which is too big to lie inside the original universe \mathbf{U} (see Remark 6.2.16), is equivalent to a \mathbf{U} -category. On the practical side, it means that among K -injective objects we do not need fractions to represent morphisms.

Corollary 9.1.18. *Let \star and \dagger be boundedness conditions such that*

$$\mathbf{K}^\star(A, M) \subseteq \mathbf{K}^\dagger(A, M).$$

Assume these categories have enough K -injectives. Then the canonical functor

$$\mathbf{D}^\star(A, M) \rightarrow \mathbf{D}^\dagger(A, M)$$

is fully faithful.

Proof. Combine Corollary 9.1.16 with the fact that $\mathbf{K}^\star(A, M) \rightarrow \mathbf{K}^\dagger(A, M)$ is fully faithful. \square

Remark 9.1.19. Earlier we only proved that $\mathbf{D}^\star(A, M) \rightarrow \mathbf{D}(A, M)$ is fully faithful in special cases (see Proposition 7.3.5 and Exercise 7.3.12).

Corollary 9.1.20. *Let $\phi : I \rightarrow J$ be a morphism in $\mathbf{C}_{\text{str}}(A, M)$ between K -injective objects. Then ϕ is a homotopy equivalence if and only if it is a quasi-isomorphism.*

Proof. One implication is trivial. For the reverse implication, if ϕ is a quasi-isomorphism then it is an isomorphism in $\mathbf{D}(A, M)$, and by Theorem 9.1.13 for $\mathbf{K} = \mathbf{K}(A, M)$ we see that ϕ is an isomorphism in $\mathbf{K}(A, M)$. \square

Here is another useful definition. It is a variant of Definition 8.3.5.

Definition 9.1.21. Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, M)$, and assume \mathbf{K} has enough K -injectives. A *system of K -injective resolutions* in \mathbf{K} is a pair (I, ρ) , where $I : \text{Ob}(\mathbf{K}) \rightarrow \text{Ob}(\mathbf{K}_{\text{inj}})$ is a function, and $\rho = \{\rho_M\}_{M \in \text{Ob}(\mathbf{K})}$ is a collection of quasi-isomorphisms $\rho_M : M \rightarrow I(M)$ in \mathbf{K} . Moreover, if $M \in \text{Ob}(\mathbf{K}_{\text{inj}})$, then $I(M) = M$ and $\rho_M = \text{id}_M$.

The proposition below is a variant of Lemma 8.3.12.

Proposition 9.1.22. *Suppose a system of K -injective resolutions (I, ρ) has been chosen. Then the function I extends uniquely to a triangulated functor $I : \mathbf{K}_S \rightarrow \mathbf{K}_{\text{inj}}$, such that $\text{Id}_{\mathbf{K}_{\text{inj}}} = I \circ Q|_{\mathbf{K}_{\text{inj}}}$, and $\rho : \text{Id}_{\mathbf{K}_S} \Rightarrow Q \circ I$ is an isomorphism of triangulated functors.*

Proof. The proof is the same as that of Lemma 8.3.12, except that here we use Corollary 9.1.15. \square

The next corollary is a categorical interpretation of the last proposition.

Corollary 9.1.23 (Functorial K -Injective Resolutions). *Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, M)$, and assume \mathbf{K} has enough K -injectives.*

- (1) *There are a triangulated functor $I : \mathbf{K} \rightarrow \mathbf{K}$ and a morphism of triangulated functors $\rho : \text{Id}_{\mathbf{K}} \rightarrow I$, such that for any object $M \in \mathbf{K}$ the object $I(M)$ is K -injective, and the morphism $\rho_M : M \rightarrow I(M)$ is a quasi-isomorphism.*
- (2) *If (I', ρ') is another such pair, then there is a unique isomorphism of triangulated functors $\zeta : I \xrightarrow{\sim} I'$ such that $\rho' = \zeta \circ \rho$.*

Exercise 9.1.24. Prove Corollary 9.1.23.

Theorem 9.1.25. *Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, M)$, and denote by S the set of quasi-isomorphisms in \mathbf{K} . Assume \mathbf{K} has enough K -injectives. Let \mathbf{E} be any pretriangulated category, and let*

$$F : \mathbf{K} \rightarrow \mathbf{E}$$

be any triangulated functor. Then F has a right derived functor

$$(RF, \eta) : \mathbf{K}_S \rightarrow \mathbf{E}.$$

Furthermore, for any $I \in \mathbf{K}_{\text{inj}}$ the morphism $\eta_I : F(I) \rightarrow RF(I)$ in \mathbf{E} is an isomorphism.

Proof. We will use Theorem 8.3.3. In the notation of that theorem, let $\mathbf{J} := \mathbf{K}_{\text{inj}}$. Condition (b) of that theorem holds (this is the “enough K-injectives” assertion). Next, Theorem 9.1.13 implies that any quasi-isomorphism $\phi : I \rightarrow J$ in \mathbf{K}_{inj} is actually an isomorphism. Therefore $F(\phi)$ is an isomorphism in \mathbf{E} , and this is condition (a) of Theorem 8.3.3. \square

Example 9.1.26. Let A be any DG ring. We will prove later that $\mathbf{K}(A)$ has enough K-injectives. Therefore, given any triangulated functor $F : \mathbf{K}(A) \rightarrow \mathbf{E}$ into any pretriangulated category \mathbf{E} , the right derived functor

$$(RF, \eta) : \mathbf{D}(A) \rightarrow \mathbf{E}$$

exists.

Suppose we choose a system of K-injective resolutions (I, ρ) in $\mathbf{K}(A)$. Then we get a presentation of (RF, η) as follows: $RF(M) = F(I(M))$ and $\eta_M = F(\rho_M)$.

9.2. K-Projective DG Modules. This subsection is dual to the previous one, and so we will be brief.

Definition 9.2.1. A DG module $P \in \mathbf{C}(A, \mathbf{M})$ is called *K-projective* if for every acyclic DG module $N \in \mathbf{C}(A, \mathbf{M})$, the DG \mathbb{K} -module $\text{Hom}_{A, \mathbf{M}}(P, N)$ is acyclic.

Proposition 9.2.2. A DG module $P \in \mathbf{K}(A, \mathbf{M})$ is K-projective if and only if $\text{Hom}_{\mathbf{K}(A, \mathbf{M})}(P, N) = 0$ for every acyclic DG module $N \in \mathbf{K}(A, \mathbf{M})$.

The proof is like that of Proposition 9.1.3.

Definition 9.2.3. Let $M \in \mathbf{K}(A, \mathbf{M})$. A *K-projective resolution* of M is a quasi-isomorphism $\rho : P \rightarrow M$ in $\mathbf{K}(A, \mathbf{M})$, where P is a K-projective DG module.

Definition 9.2.4. Let \mathbf{K} be a full subcategory of $\mathbf{K}(A, \mathbf{M})$. The full subcategory of \mathbf{K} on the K-projective DG modules in it is denoted by \mathbf{K}_{prj} . In other words,

$$\mathbf{K}_{\text{prj}} = \mathbf{K}(A, \mathbf{M})_{\text{prj}} \cap \mathbf{K}.$$

The same warning after Definition 9.1.8 applies here.

Proposition 9.2.5. If \mathbf{K} is a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$, then \mathbf{K}_{prj} is a full pretriangulated subcategory of \mathbf{K} .

The proof is like that of Proposition 9.1.9.

Example 9.2.6. Let \star be some boundedness condition (namely b , $+$, $-$ or nothing). Since $\mathbf{K}^\star(A, \mathbf{M})$ is a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$, we see that $\mathbf{K}^\star(A, \mathbf{M})_{\text{prj}}$ is a pretriangulated subcategory too.

Definition 9.2.7. Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. We say that \mathbf{K} has enough K-projectives if any DG module $M \in \mathbf{K}$ admits a K-projective resolution inside \mathbf{K} . I.e. there is a quasi-isomorphism $\rho : P \rightarrow M$ where $P \in \mathbf{K}_{\text{prj}}$.

Lemma 9.2.8. Let \mathbf{K} be a full subcategory of $\mathbf{K}(A, \mathbf{M})$. Let $s : M \rightarrow P$ be a quasi-isomorphism in \mathbf{K} , and assume P is K-projective. Then s has a right inverse; namely there is a morphism $t : P \rightarrow M$ in \mathbf{K} such that $s \circ t = \text{id}_P$.

Same proof as that of Lemma 9.1.12.

Theorem 9.2.9. *Let A be a DG ring, let \mathbf{M} be an abelian category, and let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. Denote by \mathbf{S} the set of quasi-isomorphisms in \mathbf{K} . Then the localization functor*

$$Q : \mathbf{K}_{\text{prj}} \rightarrow \mathbf{K}_{\mathbf{S}}$$

is fully faithful.

The proof is the same as that of Theorem 9.1.13, with reversed arrow. The next corollaries and proposition are also proved like their \mathbf{K} -injective counterparts.

Corollary 9.2.10. *Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$. If \mathbf{K} has enough K -projectives, then the localization functor*

$$Q : \mathbf{K}_{\text{prj}} \rightarrow \mathbf{K}_{\mathbf{S}}$$

is an equivalence.

Corollary 9.2.11. *Let \star and \dagger be boundedness conditions such that*

$$\mathbf{K}^{\star}(A, \mathbf{M}) \subseteq \mathbf{K}^{\dagger}(A, \mathbf{M}).$$

Assume these categories have enough K -projectives. Then the canonical functor

$$\mathbf{D}^{\star}(A, \mathbf{M}) \rightarrow \mathbf{D}^{\dagger}(A, \mathbf{M})$$

is fully faithful.

Corollary 9.2.12. *Let $\phi : P \rightarrow Q$ be a morphism in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$ between K -projective objects. Then ϕ is a homotopy equivalence if and only if it is a quasi-isomorphism.*

Definition 9.2.13. Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$, and assume \mathbf{K} has enough K -projectives. A *system of K -projective resolutions* in \mathbf{K} is a pair (P, ρ) , where $P : \text{Ob}(\mathbf{K}) \rightarrow \text{Ob}(\mathbf{K}_{\text{prj}})$ is a function, and $\rho = \{\rho_M\}_{M \in \text{Ob}(\mathbf{K})}$ is a collection of quasi-isomorphisms $\rho_M : P(M) \rightarrow M$ in \mathbf{K} . Moreover, if $M \in \text{Ob}(\mathbf{K}_{\text{prj}})$, then $P(M) = M$ and $\rho_M = \text{id}_M$.

Proposition 9.2.14. *Suppose a system of K -projective resolutions (P, ρ) has been chosen. Then the function P extends uniquely to a triangulated functor $P : \mathbf{K}_{\mathbf{S}} \rightarrow \mathbf{K}_{\text{prj}}$, such that $\text{Id}_{\mathbf{K}_{\text{prj}}} = P \circ Q|_{\mathbf{K}_{\text{prj}}}$, and $\rho : Q \circ P \Rightarrow \text{Id}_{\mathbf{K}_{\mathbf{S}}}$ is an isomorphism of triangulated functors.*

Corollary 9.2.15 (Functorial K -Projective Resolutions). *Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$, and assume \mathbf{K} has enough K -projectives.*

- (1) *There are a triangulated functor $P : \mathbf{K} \rightarrow \mathbf{K}$ and a morphism of triangulated functors $\rho : P \rightarrow \text{Id}_{\mathbf{K}}$, such that for any object $M \in \mathbf{K}$ the object $P(M)$ is K -projective, and the morphism $\rho_M : P(M) \rightarrow M$ is a quasi-isomorphism.*
- (2) *If (P', ρ') is another such pair, then there is a unique isomorphism of triangulated functors $\zeta : P' \xrightarrow{\sim} P$ such that $\rho' = \rho \circ \zeta$.*

Theorem 9.2.16. *Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A, \mathbf{M})$, and denote by \mathbf{S} the set of quasi-isomorphisms in \mathbf{K} . Assume \mathbf{K} has enough K -projectives. Let \mathbf{E} be any pretriangulated category, and let*

$$F : \mathbf{K} \rightarrow \mathbf{E}$$

be any triangulated functor. Then F has a left derived functor

$$(LF, \eta) : \mathbf{K}_S \rightarrow \mathbf{E}.$$

Furthermore, for any $P \in \mathbf{K}_{\text{prj}}$ the morphism $\eta_P : LF(P) \rightarrow F(P)$ in \mathbf{E} is an isomorphism.

The proof is like that of Theorem 9.1.25.

Example 9.2.17. Let A be any DG ring. We will prove later that $\mathbf{K}(A)$ has enough \mathbf{K} -projectives. Therefore, given any triangulated functor $F : \mathbf{K}(A) \rightarrow \mathbf{E}$ into any pretriangulated category \mathbf{E} , the left derived functor

$$(LF, \eta) : \mathbf{D}(A) \rightarrow \mathbf{E}$$

exists.

Suppose we choose a system of \mathbf{K} -projective resolutions (P, ρ) in $\mathbf{K}(A)$. Then we get a presentation of (LF, η) as follows: $LF(M) = F(P(M))$ and $\eta_M = F(\rho_M)$.

9.3. \mathbf{K} -Flat DG Modules. Recall that A^{op} is the opposite DG ring. The objects of $\mathbf{C}(A^{\text{op}})$ are the right DG A -modules.

Definition 9.3.1. A DG module $P \in \mathbf{C}(A)$ is called *K-flat* if for every acyclic DG module $N \in \mathbf{C}(A^{\text{op}})$, the DG \mathbb{K} -module $N \otimes_A P$ is acyclic.

Proposition 9.3.2. *If $P \in \mathbf{C}(A)$ is \mathbf{K} -projective then it is \mathbf{K} -flat.*

Proof. Let \mathbb{K}^* be an injective cogenerator of $\mathbf{M}(\mathbb{K}) = \text{Mod } \mathbb{K}$. This means that \mathbb{K}^* is an injective \mathbb{K} -module, such that any nonzero \mathbb{K} -module W admits a nonzero homomorphism $W \rightarrow \mathbb{K}^*$. A universal choice is $\mathbb{K}^* = \text{Hom}_{\mathbb{Z}}(\mathbb{K}, \mathbb{Q}/\mathbb{Z})$. It is not hard to see that a DG \mathbb{K} -module W is acyclic if and only if $\text{Hom}_{\mathbb{K}}(W, \mathbb{K}^*)$ is acyclic. (Cf. Exercise 10.5.6 for a stronger assertion.)

Take an acyclic complex $N \in \mathbf{C}(A^{\text{op}})$. Then by Hom-tensor adjunction there is an isomorphism of DG \mathbb{K} -modules

$$\text{Hom}_{\mathbb{K}}(N \otimes_A P, \mathbb{K}^*) \cong \text{Hom}_A(P, \text{Hom}_{\mathbb{K}}(N, \mathbb{K}^*)).$$

The right side is acyclic by our assumptions. Hence so is the left side. It follows that $N \otimes_A P$ is acyclic. \square

The proof above also gives a hint to the next proposition.

Proposition 9.3.3. *A DG module $P \in \mathbf{K}(A)$ is \mathbf{K} -flat iff*

$$\text{Hom}_{\mathbf{K}(A)}(P, \text{Hom}_{\mathbb{K}}(N, J)) = 0$$

for every acyclic $N \in \mathbf{C}(A^{\text{op}})$ and every injective $J \in \text{Mod } \mathbb{K}$.

Exercise 9.3.4. Prove Proposition 9.3.3.

The next proposition will be subsumed later, in Section 11.5, in a theorem about the left derived tensor bifunctor.

Proposition 9.3.5. *Let \mathbf{K} be a full pretriangulated subcategory of $\mathbf{K}(A)$, and denote by S the set of quasi-isomorphisms in \mathbf{K} . Assume \mathbf{K} has enough \mathbf{K} -flat objects. Let B be another central DG \mathbb{K} -ring, let $N \in \mathbf{K}(B \otimes_{\mathbb{K}} A^{\text{op}})$, and define*

$$F : \mathbf{K} \rightarrow \mathbf{D}(B)$$

to be the triangulated functor $F(M) := Q(N \otimes_A M)$, as in Example 4.4.6 and Theorem 5.4.13. Then F has a left derived functor

$$(LF, \eta) : K_S \rightarrow \mathbf{D}(B).$$

Furthermore, for any object $P \in K$ which is K -flat, the morphism $\eta_P : LF(P) \rightarrow F(P)$ in $\mathbf{D}(B)$ is an isomorphism.

Exercise 9.3.6. Prove Proposition 9.3.5. (Hint: look at the proof of Theorem 9.1.25.)

Remark 9.3.7. In view of Proposition 9.3.2, the reader might wonder why we bother with K -flat DG modules. The reason is that on a ringed space (X, \mathcal{A}) there are usually very few projective \mathcal{A} -modules. But, as we shall prove, there are enough K -flat complexes in $\mathbf{C}(\mathcal{A}) = \mathbf{C}(\text{Mod } \mathcal{A})$. This will allow us to have a left derived tensor functor for sheaves.

10. EXISTENCE OF RESOLUTIONS

In this section we continue in the more concrete setting: A is a DG ring, and \mathcal{M} is an abelian category (both over a commutative base ring \mathbb{K}). We will prove existence of K-projective and K-injective resolutions in several contexts.

10.1. Direct and Inverse Limits of Complexes. We shall have to work with limits in this section. Limits in abstract abelian and DG categories (not to mention pretriangulated categories) are a very delicate issue. We will try to be as concrete as possible, in order to avoid pitfalls and confusion.

Let \mathcal{C} be an arbitrary category (not necessarily linear). A *direct system* in \mathcal{C} is data

$$(\{M_k\}_{k \in \mathbb{N}}, \{\mu_k\}_{k \in \mathbb{N}}),$$

where M_k are objects of \mathcal{C} , and $\mu_k : M_k \rightarrow M_{k+1}$ are morphisms, called transitions. The *direct limit*

$$M = \lim_{k \rightarrow} M_k$$

need not exist in \mathcal{C} ; but if it does, then it is unique up to a unique isomorphism.

By an *inverse system* in the category \mathcal{C} we mean data

$$(\{M_k\}_{k \in \mathbb{N}}, \{\mu_k\}_{k \in \mathbb{N}}),$$

where $\{M_k\}_{k \in \mathbb{N}}$ is a collection of objects, and $\mu_k : M_{k+1} \rightarrow M_k$ are morphisms, also called transitions. The *inverse limit*

$$M = \lim_{\leftarrow k} M_k$$

need not exist in \mathcal{C} ; but if it does, then it is unique up to a unique isomorphism.

Proposition 10.1.1 (Sandwiched Systems). *Let \mathcal{C} be a category.*

- (1) *Let $(\{M_k\}_{k \in \mathbb{N}}, \{\mu_k\}_{k \in \mathbb{N}})$ and $(\{N_k\}_{k \in \mathbb{N}}, \{\nu_k\}_{k \in \mathbb{N}})$ be direct systems in \mathcal{C} . Assume there are morphisms $\alpha_k : M_k \rightarrow N_k$ and $\beta_k : N_k \rightarrow M_{k+1}$, such that $\beta_k \circ \alpha_k = \mu_k$ and $\alpha_{k+1} \circ \beta_k = \nu_k$ for all k . If the limit $N = \lim_{k \rightarrow} N_k$ exists, then the limit $M = \lim_{k \rightarrow} M_k$ also exists, and the canonical morphism $\alpha : M \rightarrow N$ is an isomorphism.*
- (2) *Let $(\{M_k\}_{k \in \mathbb{N}}, \{\mu_k\}_{k \in \mathbb{N}})$ and $(\{N_k\}_{k \in \mathbb{N}}, \{\nu_k\}_{k \in \mathbb{N}})$ be inverse systems in \mathcal{C} . Assume there are morphisms $\alpha_k : M_k \rightarrow N_k$ and $\beta_k : N_k \rightarrow M_{k-1}$, such that $\beta_k \circ \alpha_k = \mu_{k-1}$ and $\alpha_{k-1} \circ \beta_k = \nu_{k-1}$ for all k . If the limit $N = \lim_{\leftarrow k} N_k$ exists, then the limit $M = \lim_{\leftarrow k} M_k$ also exists, and the canonical morphism $\alpha : M \rightarrow N$ is an isomorphism.*

In other words, sandwiched systems behave the same regarding limits. The direct systems (item (1)) look like this:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_k & \xrightarrow{\alpha_k} & N_k & \xrightarrow{\beta_k} & M_{k+1} & \xrightarrow{\alpha_{k+1}} & N_{k+1} & \xrightarrow{\beta_{k+1}} & M_{k+2} & \longrightarrow & \cdots \\ & & & & & \text{curved arrow } \mu_k & & & \text{curved arrow } \mu_{k+1} & & & & \\ & & & & & & & & & & & & \end{array}$$

Exercise 10.1.2. Prove Proposition 10.1.1.

Proposition 10.1.3. *Let \mathcal{C} be a category.*

- (1) Let $\{M_k\}_{k \in \mathbb{N}}$ be a direct system in \mathcal{C} , and assume the direct limit $M = \lim_{k \rightarrow} M_k$ exists. Then for any object $N \in \mathcal{C}$, the canonical function

$$\mathrm{Hom}_{\mathcal{C}}(M, N) \rightarrow \lim_{\leftarrow k} \mathrm{Hom}_{\mathcal{C}}(M_k, N)$$

is bijective.

- (2) Let $\{M_k\}_{k \in \mathbb{N}}$ be an inverse system in \mathcal{C} , and assume the inverse limit $M = \lim_{\leftarrow k} M_k$ exists. Then for any object $N \in \mathcal{C}$, the canonical function

$$\mathrm{Hom}_{\mathcal{C}}(N, M) \rightarrow \lim_{\leftarrow k} \mathrm{Hom}_{\mathcal{C}}(N, M_k)$$

is bijective.

Exercise 10.1.4. Prove Proposition 10.1.3.

Now we start talking about limits in the abelian category $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$. We have to be careful, because it is often not true that limits exist in abelian categories.

Example 10.1.5. Let \mathbf{M} be the category of finite abelian groups. The inverse system $\{M_k\}_{k \in \mathbb{N}}$, where $M_k := \mathbb{Z}/(2^k)$, and the transition $\mu_k : M_{k+1} \rightarrow M_k$ is the canonical surjection, does not have an inverse limit in \mathbf{M} . We can also make $\{M_k\}_{k \in \mathbb{N}}$ into a direct system, in which the transition $\nu_k : M_k \rightarrow M_{k+1}$ is multiplication by 2. The direct limit does not exist in \mathbf{M} .

Proposition 10.1.6.

- (1) Let $\{M_k\}_{k \in \mathbb{N}}$ be a direct system in $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$. Assume that for every i the direct limit $\lim_{k \rightarrow} M_k^i$ exists in \mathbf{M} . Then the direct limit $M = \lim_{k \rightarrow} M_k$ exists in $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$, and in degree i it is $M^i = \lim_{k \rightarrow} M_k^i$.
- (2) Let $\{M_k\}_{k \in \mathbb{N}}$ be an inverse system in $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$. Assume that for every i the inverse limit $\lim_{\leftarrow k} M_k^i$ exists in \mathbf{M} . Then the inverse limit $M = \lim_{\leftarrow k} M_k$ exists in $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$, and in degree i it is $M^i = \lim_{\leftarrow k} M_k^i$.

Proof. We will only prove item (1); the proof of item (2) is identical. For any integer i define $M^i := \lim_{k \rightarrow} M_k^i \in \mathbf{M}$. By the universal property of the direct limit, the differentials $d : M_k^i \rightarrow M_k^{i+1}$ induce differentials $d : M^i \rightarrow M^{i+1}$, and in this way we obtain a complex $M := \{M^i\}_{i \in \mathbb{Z}} \in \mathbf{C}(\mathbf{M})$. Similarly, any element $a \in A^j$ induces morphisms $a : M^i \rightarrow M^{i+j}$ in \mathbf{M} , and thus M becomes an object of $\mathbf{C}(A, \mathbf{M})$. There are morphisms $M_k \rightarrow M$ in $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$, and it is easy to see that these make M into a direct limit of the system $\{M_k\}_{k \in \mathbb{N}}$. \square

Since limits exist in $\mathbf{M} = \mathrm{Mod} \mathbb{K}$, the proposition above says that they exist in $\mathbf{C}(A)$. Similarly they exist in the category $\mathbf{G}(\mathbb{K})$ of graded \mathbb{K} -modules.

We say that a direct system $\{M_k\}_{k \in \mathbb{N}}$ in \mathbf{M} is *eventually stationary* if $\mu_k : M_k \rightarrow M_{k+1}$ are isomorphisms for large k . Similarly we can talk about an eventually stationary inverse system. The limit of an eventually stationary system (direct or inverse) always exists: it is M_k for large enough k .

Proposition 10.1.7.

- (1) Let $\{M_k\}_{k \in \mathbb{N}}$ be a direct system in $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$. Assume that for each i the direct system $\{M_k^i\}_{k \in \mathbb{N}}$ in \mathbf{M} is eventually stationary. Then the direct limit $M = \lim_{k \rightarrow} M_k$ exists in $\mathbf{C}_{\mathrm{str}}(A, \mathbf{M})$, the direct limit $\lim_{k \rightarrow} H(M_k)$ exists in $\mathbf{G}^0(\mathbf{M})$, and the canonical morphism

$$\lim_{k \rightarrow} H(M_k) \rightarrow H(M)$$

in $\mathbf{G}^0(\mathbf{M})$ is an isomorphism.

- (2) Let $\{M_k\}_{k \in \mathbb{N}}$ be an inverse system in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$. Assume that for each i the inverse system $\{M_k^i\}_{k \in \mathbb{N}}$ in \mathbf{M} is eventually stationary. Then the inverse limit $M = \lim_{\leftarrow k} M_k$ exists in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$, the inverse limit $\lim_{\leftarrow k} H(M_k)$ exists in $\mathbf{G}^0(\mathbf{M})$, and the canonical morphism

$$H(M) \rightarrow \lim_{\leftarrow k} H(M_k)$$

in $\mathbf{G}^0(\mathbf{M})$ is an isomorphism.

Proof. (1) As mentioned above, for each i the limit $M^i = \lim_{k \rightarrow} M_k^i$ exists in \mathbf{M} . By Proposition 10.1.6 the limit $M = \lim_{k \rightarrow} M_k$ exists in $\mathbf{C}_{\text{str}}(A, \mathbf{M})$.

Regarding the cohomology: fix an integer i . Take k large enough such that $M_k^{i'} \rightarrow M_{k'}^{i'}$ are isomorphisms for all $k \leq k'$ and $i-1 \leq i' \leq i+1$. Then $M_k^{i'} \rightarrow M_{k'}^{i'}$ are isomorphisms in this range, and therefore $H^i(M_{k'}) \rightarrow H^i(M)$ are isomorphisms for all $k \leq k'$. We see that the direct system $\{H^i(M_k)\}_{k \in \mathbb{N}}$ is eventually stationary, and its direct limit is $H^i(M)$.

- (2) The same. □

When we drop the abstract abelian category \mathbf{M} , i.e. when we work with $\mathbf{M} = \text{Mod } \mathbb{K} = \mathbf{M}(\mathbb{K})$ and $\mathbf{C}_{\text{str}}(A, \mathbf{M}) = \mathbf{C}_{\text{str}}(A)$, there is no problem of existence of limits. The next proposition says that furthermore “direct limits are exact” in $\mathbf{C}_{\text{str}}(A)$.

Proposition 10.1.8. *Let $\{M_k\}_{k \in \mathbb{N}}$ be a direct system in $\mathbf{C}_{\text{str}}(A)$. Then the canonical homomorphism*

$$\lim_{k \rightarrow} H(M_k) \rightarrow H(M)$$

in $\mathbf{G}^0(\mathbb{K})$ is bijective.

Exercise 10.1.9. Prove Proposition 10.1.8. (Hint: forget the action of A , and work with complexes of abelian groups.)

Exactness of inverse limits tends to be much more complicated than that of direct limits, even for \mathbb{K} -modules. We always have to make some condition on the inverse system to have exactness in the limit.

Definition 10.1.10. Let $(\{M_k\}_{k \in \mathbb{N}}, \{\mu_k\}_{k \in \mathbb{N}})$ be an inverse system in $\mathbf{M}(\mathbb{K})$. For any $l \geq k$ let $M_{l,k} \subseteq M_k$ be the image of the homomorphism

$$\text{id} \circ \mu_k \circ \cdots \circ \mu_{l-1} : M_l \rightarrow M_k.$$

Note that there are inclusions $M_{l+1,k} \subseteq M_{l,k}$, so for fixed k we have an inverse system $\{M_{l,k}\}_{l \geq k}$.

We say that the inverse system $\{M_k\}_{k \in \mathbb{N}}$ has the *Mittag-Leffler* property if for every index k , the inverse system $\{M_{l,k}\}_{l \geq k}$ is eventually stationary.

Example 10.1.11. If the system

$$(\{M_k\}_{k \in \mathbb{N}}, \{\mu_k\}_{k \in \mathbb{N}})$$

satisfies one of the following conditions, then it has the Mittag-Leffler property:

- (a) The system has surjective transitions.
- (b) The system is eventually stationary.
- (c) For any $k \in \mathbb{N}$ there exists some $l \geq k$ such that $M_{l,k} = 0$. This is called the *trivial Mittag-Leffler property*, and one says that the system is *pro-zero*.

Theorem 10.1.12 (Mittag-Leffler Argument). *Let $\{M_k\}_{k \in \mathbb{N}}$ be an inverse system in $\mathbf{C}_{\text{str}}(A)$, with inverse limit $M = \lim_{\leftarrow k} M_k$. Assume the system satisfies these two conditions:*

- (a) *For every $i \in \mathbb{Z}$ the inverse system $\{M_k^i\}_{k \in \mathbb{N}}$ in $\mathbf{M}(\mathbb{K})$ has the Mittag-Leffler property.*
- (b) *For every $i \in \mathbb{Z}$ the inverse system $\{H^i(M_k)\}_{k \in \mathbb{N}}$ in $\mathbf{M}(\mathbb{K})$ has the Mittag-Leffler property.*

Then the canonical homomorphisms

$$H^i(M) \rightarrow \lim_{\leftarrow k} H^i(M_k)$$

are bijective.

Proof. We can forget all about the graded A -module structure, and just view this as an inverse system in $\mathbf{C}_{\text{str}}(\mathbb{Z})$, i.e. and inverse system of complexes of abelian groups. Now this is a special case of [KaSc1, Proposition 1.12.4] or [EGA III, Ch. 0_{III}, Proposition 13.2.3]. \square

The most useful instance of the ML argument is this:

Corollary 10.1.13. *Let $\{M_k\}_{k \in \mathbb{N}}$ be an inverse system in $\mathbf{C}_{\text{str}}(A)$, with inverse limit $M = \lim_{\leftarrow k} M_k$. Assume the system satisfies these two conditions:*

- (a) *For every $i \in \mathbb{Z}$ the inverse system $\{M_k^i\}_{k \in \mathbb{N}}$ has surjective transitions.*
- (b) *For every k the DG module M_k is acyclic.*

Then M is acyclic.

Proof. Conditions (a) and (b) here imply conditions (a) and (b) of Theorem 10.1.12, respectively. \square

Exercise 10.1.14. Prove Corollary 10.1.13 directly, without resorting to Theorem 10.1.12.

Remark 10.1.15. We will not attempt discussing direct or inverse limits in abstract abelian categories. Such definitions do exist (e.g. for a *Grothendieck abelian category*, cf. [KaSc2, Definition 8.3.24]), but this sort of thing is a source of anxiety (and sometimes of errors).

Before going on, it is good to remember the roles of the objects of cocycles and coboundaries. Let $M \in \mathbf{C}(A, \mathbf{M})$. The object of coboundaries $Z(M) \subseteq M$ is defined by

$$Z^i(M) := \text{Ker}(d : M^i \rightarrow M^{i+1}).$$

The object of cocycles $B(M) \subseteq M$ is defined by

$$B^i(M) := \text{Im}(d : M^{i-1} \rightarrow M^i).$$

Note that $Z(A)$ is a DG ring with trivial differential, and the objects $Z(M)$ and $B(M)$ live in $\mathbf{C}(Z(A), \mathbf{M})$, with trivial differentials too. There are exact sequences

$$(10.1.16) \quad 0 \rightarrow Z(M) \rightarrow M \xrightarrow{d} T(B(M)) \rightarrow 0$$

and

$$(10.1.17) \quad 0 \rightarrow B(M) \rightarrow Z(M) \rightarrow H(M) \rightarrow 0$$

in $\mathbf{C}_{\text{str}}(Z(A), \mathbf{M})$.

10.2. K-Projective Resolutions in $\mathbf{C}^-(\mathbf{M})$. Recall that \mathbf{M} is some abelian category, and $\mathbf{C}(\mathbf{M})$ is the DG category of complexes in \mathbf{M} . The strict category $\mathbf{C}_{\text{str}}(\mathbf{M})$ is abelian.

A *filtration* on a complex $M \in \mathbf{C}_{\text{str}}(\mathbf{M})$ is a collection $\{F_j(M)\}_{j \geq -1}$ of subobjects of M , such that $F_j(M) \subseteq F_{j+1}(M)$. This is a particular kind of direct system in $\mathbf{C}_{\text{str}}(\mathbf{M})$. We say that $M = \lim_{j \rightarrow} F_j(M)$ if this limit exists in $\mathbf{C}_{\text{str}}(\mathbf{M})$, and the canonical morphism $\lim_{j \rightarrow} F_j(M) \rightarrow M$ is an isomorphism. There are also the subquotients

$$(10.2.1) \quad \text{gr}_j^F(M) := F_j(M)/F_{j-1}(M) \in \mathbf{C}_{\text{str}}(\mathbf{M})$$

for $j \geq 0$. Sometimes we will be interested in filtrations that have finite length, by which we mean a direct system of subobjects $\{F_j(M)\}_{-1 \leq j \leq k}$ for some $k < \infty$. In this case $\text{gr}_j^F(M)$ is defined only for $0 \leq j \leq k$.

The next definition is inspired by the work of Keller [Ke, Section 3.1].

Definition 10.2.2. Let P be an object of $\mathbf{C}(\mathbf{M})$.

- (1) A *semi-projective filtration* on P is a filtration $F = \{F_j(P)\}_{j \geq -1}$ on P as an object of $\mathbf{C}_{\text{str}}(\mathbf{M})$, such that:
 - $F_{-1}(P) = 0$.
 - Each $\text{gr}_j^F(P)$ is a complex of projective objects of \mathbf{M} with zero differential.
 - $P = \lim_{j \rightarrow} F_j(P)$ in $\mathbf{C}_{\text{str}}(\mathbf{M})$.
- (2) The complex P is called a *semi-projective complex* if it admits some semi-projective filtration.

Theorem 10.2.3. Let \mathbf{M} be an abelian category, and let P be a semi-projective complex in $\mathbf{C}(\mathbf{M})$. Then P is K -projective.

Proof. Step 1. We start by proving that if $P = T^k(Q)$, the translation of a projective object $Q \in \mathbf{M}$, then P is K -projective. This is easy: given an acyclic complex $N \in \mathbf{C}(\mathbf{M})$, we have

$$\text{Hom}_{\mathbf{M}}(P, N) = \text{Hom}_{\mathbf{M}}(T^k(Q), N) \cong T^{-k}(\text{Hom}_{\mathbf{M}}(Q, N))$$

in $\mathbf{C}_{\text{str}}(\mathbb{K})$. But $\text{Hom}_{\mathbf{M}}(Q, -)$ is an exact functor $\mathbf{M} \rightarrow \mathbf{M}(\mathbb{K})$, so $\text{Hom}_{\mathbf{M}}(Q, N)$ is an acyclic complex.

Step 2. Now P is a complex of projective objects of \mathbf{M} with zero differential. This means that

$$P \cong \bigoplus_{k \in \mathbb{Z}} T^k(Q_k)$$

in $\mathbf{C}_{\text{str}}(\mathbf{M})$, where each Q_k is a projective object in \mathbf{M} . But then

$$\text{Hom}_{\mathbf{M}}(P, N) \cong \prod_{k \in \mathbb{Z}} \text{Hom}_{\mathbf{M}}(T^k(Q_k), N).$$

This is an easy case of Proposition 10.1.3. By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}_{\text{str}}(\mathbb{K})$ is acyclic (itself an easy case of the Mittag-Leffler argument), we conclude that $\text{Hom}_{\mathbf{M}}(P, N)$ is acyclic.

Step 3. Fix a semi-projective filtration $F = \{F_j(P)\}_{j \geq -1}$ on P . Here we prove that for every j the complex $F_j(P)$ is K -projective. This is done by induction on $j \geq -1$. For $j = -1$ it is trivial. For $j \geq 0$ there is an exact sequence of complexes

$$(10.2.4) \quad 0 \rightarrow F_{j-1}(P) \rightarrow F_j(P) \rightarrow \text{gr}_j^F(P) \rightarrow 0$$

in $\mathbf{C}(\mathbf{M})$. In each degree $i \in \mathbb{Z}$ the exact sequence

$$0 \rightarrow F_{j-1}(P)^i \rightarrow F_j(P)^i \rightarrow \mathrm{gr}_j^F(P)^i \rightarrow 0$$

in \mathbf{M} splits, because $\mathrm{gr}_j^F(P)^i$ is a projective object. Thus the exact sequence (10.2.4) is split exact in the abelian category $\mathbf{G}^0(\mathbf{M})$ of graded objects in \mathbf{M} .

Let $N \in \mathbf{C}(\mathbf{M})$ be an acyclic complex. Applying the functor $\mathrm{Hom}_{\mathbf{M}}(-, N)$ to the sequence of complexes (10.2.4) we obtain a sequence (10.2.5)

$$0 \rightarrow \mathrm{Hom}_{\mathbf{M}}(\mathrm{gr}_j^F(P), N) \rightarrow \mathrm{Hom}_{\mathbf{M}}(F_j(P), N) \rightarrow \mathrm{Hom}_{\mathbf{M}}(F_{j-1}(P), N) \rightarrow 0$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. Because (10.2.4) is split exact in $\mathbf{G}^0(\mathbf{M})$, the sequence (10.2.5) is split exact in $\mathbf{G}^0(\mathbb{K})$. Therefore (10.2.5) is exact in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the complex $\mathrm{Hom}_{\mathbf{M}}(F_{j-1}(P), N)$ is acyclic. By step 1 the complex $\mathrm{Hom}_{\mathbf{M}}(\mathrm{gr}_j^F(P), N)$ is acyclic. The long exact cohomology sequence associated to (10.2.5) shows that the complex $\mathrm{Hom}_{\mathbf{M}}(F_j(P), N)$ is acyclic too.

Step 4. We keep the semi-projective filtration $F = \{F_j(P)\}_{j \geq -1}$ from step 3. Take any acyclic complex $N \in \mathbf{C}(\mathbf{M})$. By Proposition 10.1.3 we know that

$$\mathrm{Hom}_{\mathbf{M}}(P, N) \cong \varprojlim_j \mathrm{Hom}_{\mathbf{M}}(F_j(P), N)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_{\mathbf{M}}(F_j(P), N)$ are all acyclic. The exactness of the sequences (10.2.5) implies that the inverse system

$$\{\mathrm{Hom}_{\mathbf{M}}(F_j(P), N)\}_{j \geq -1}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\mathrm{Hom}_{\mathbf{M}}(P, N)$ is acyclic. \square

Proposition 10.2.6. *Let \mathbf{M} be an abelian category. If $P \in \mathbf{C}(\mathbf{M})$ is a bounded above complex of projectives, then P is a semi-projective complex.*

Proof. Say P is nonzero and $\sup(P) = i_1 \in \mathbb{Z}$. For $j \geq -1$ define

$$F_j(P) := (\cdots \rightarrow 0 \rightarrow P^{i_1-j} \rightarrow \cdots \rightarrow P^{i_1-1} \rightarrow P^{i_1} \rightarrow \cdots) \subseteq P.$$

Then $\{F_j(P)\}_{j \geq -1}$ is a semi-projective filtration on P . \square

The next theorem is dual to [RD, Lemma 4.6(1)], in the sense of changing injectives to projectives. (See Theorem 10.4.7 for the injective case.) We give a much more detailed proof.

Theorem 10.2.7. *Let \mathbf{M} be an abelian category with enough projectives. Any complex $M \in \mathbf{C}^-(\mathbf{M})$ admits a quasi-isomorphism $\rho : P \rightarrow M$ in $\mathbf{C}_{\mathrm{str}}^-(\mathbf{M})$, where P is a bounded above complex of projectives.*

Proof. After translating M , we can assume that $M^i = 0$ for all $i > 0$. The differential of the complex M is $d_M^i : M^i \rightarrow M^{i+1}$.

We start by choosing an epimorphism $\rho^0 : P^0 \twoheadrightarrow M^0$ in \mathbf{M} from some projective object P^0 . We get a morphism

$$\delta^0 : M^{-1} \oplus P^0 \rightarrow M^0$$

whose components are d_M^{-1} and ρ^0 . Next we choose an epimorphism

$$\psi^{-1} : P^{-1} \twoheadrightarrow \mathrm{Ker}(\delta^0)$$

from some projective object P^{-1} . So there is an exact sequence

$$P^{-1} \xrightarrow{\psi^{-1}} M^{-1} \oplus P^0 \xrightarrow{\delta^0} M^0 \rightarrow 0.$$

The components of ψ^{-1} are denoted by $\rho^{-1} : P^{-1} \rightarrow M^{-1}$ and $d_P^{-1} : P^{-1} \rightarrow P^0$.

Now to the inductive step. Here $i \leq -1$, and we already have objects P^i, \dots, P^0 , and morphisms ρ^i, \dots, ρ^0 and d_P^i, \dots, d_P^{-1} , that fit into this diagram

$$(10.2.8) \quad \begin{array}{ccccccc} P^i & \xrightarrow{d_P^i} & P^{i+1} & \longrightarrow & \dots & \xrightarrow{d_P^0} & P^0 \longrightarrow 0 \\ \downarrow \rho^i & & \downarrow \rho^{i+1} & & & & \downarrow \rho^0 \\ M^{i-1} & \xrightarrow{d_M^{i-1}} & M^i & \xrightarrow{d_M^i} & M^{i+1} & \longrightarrow & \dots \xrightarrow{d_M^0} M^0 \longrightarrow 0 \end{array}$$

in \mathcal{M} . We still did not prove this diagram is commutative.

Define the morphism

$$\delta^i : M^{i-1} \oplus P^i \rightarrow M^i \oplus P^{i+1}$$

to be the one with components $-d_M^{i-1}$, ρ^i and d_P^i . Expressing direct sums of objects as columns, and letting matrices of morphisms act on them from the left, we have this representation of δ^i :

$$(10.2.9) \quad \delta^i = \begin{bmatrix} -d_M^{i-1} & \rho^i \\ 0 & d_P^i \end{bmatrix}.$$

Let us choose an epimorphism

$$\psi^{i-1} : P^{i-1} \twoheadrightarrow \text{Ker}(\delta^i)$$

from a projective object P^{i-1} . We get an exact sequence

$$(10.2.10) \quad P^{i-1} \xrightarrow{\psi^{i-1}} M^{i-1} \oplus P^i \xrightarrow{\delta^i} M^i \oplus P^{i+1}.$$

The components of the morphism ψ^{i-1} are denoted by $\rho^{i-1} : P^{i-1} \rightarrow M^{i-1}$ and $d_P^{i-1} : P^{i-1} \rightarrow P^i$. In a matrix representation:

$$\psi^{i-1} = \begin{bmatrix} \rho^{i-1} \\ d_P^{i-1} \end{bmatrix}.$$

In this way we obtain the slightly bigger diagram

$$(10.2.11) \quad \begin{array}{ccccccc} P^{i-1} & \xrightarrow{d_P^{i-1}} & P^i & \xrightarrow{d_P^i} & P^{i+1} & \longrightarrow & \dots \longrightarrow P^0 \longrightarrow 0 \\ \downarrow \rho^{i-1} & & \downarrow \rho^i & & \downarrow \rho^{i+1} & & \downarrow \rho^0 \\ M^{i-1} & \xrightarrow{d_M^{i-1}} & M^i & \xrightarrow{d_M^i} & M^{i+1} & \longrightarrow & \dots \longrightarrow M^0 \longrightarrow 0 \end{array}$$

We carry out this construction inductively for all $i \leq -1$, thus obtaining a diagram like (10.2.11) that goes infinitely to the left.

Because $\delta^i \circ \psi^{i-1} = 0$ in (10.2.10), it follows that $d_P^i \circ d_P^{i-1} = 0$. Letting $P^i := 0$ for positive i , the collection $P := \{P^i\}_{i \in \mathbb{Z}}$ becomes a complex, with differential $d_P := \{d_P^i\}_{i \in \mathbb{Z}}$. The equality $\delta^i \circ \psi^{i-1} = 0$ also implies that

$$(10.2.12) \quad \rho^i \circ d_P^{i-1} = d_M^{i-1} \circ \rho^{i-1},$$

so the collection $\rho := \{\rho^i\}_{i \in \mathbb{Z}}$ is a strict morphism of complexes $\rho : P \rightarrow M$.

Let us examine this commutative diagram:

$$(10.2.13) \quad \begin{array}{ccccc} P^{i-1} & \xrightarrow{\psi^{i-1}} & M^{i-1} \oplus P^i & \xrightarrow{\delta^i} & M^i \oplus P^{i+1} \\ \downarrow (0, \text{id}) & & \downarrow \text{id} & & \downarrow \text{id} \\ M^{i-2} \oplus P^{i-1} & \xrightarrow{\delta^{i-1}} & M^{i-1} \oplus P^i & \xrightarrow{\delta^i} & M^i \oplus P^{i+1} \end{array}$$

The top row is exact, because it is (10.2.10). An easy calculation using (10.2.12) shows that $\delta^i \circ \delta^{i-1} = 0$. These two facts combine prove that the bottom row is also exact.

Let $N = \{N^i\}_{i \in \mathbb{Z}}$ be the complex with components $N^i := M^{i-1} \oplus P^i$ for $i \leq -1$, $N^0 := M^0$ and $N^i := 0$ for $i > 0$. The differential $d_N = \{d_N^i\}_{i \in \mathbb{Z}}$ is

$$d_N^i := \delta^i : N^i \rightarrow N^{i+1}.$$

As we saw in the paragraph above, the complex N is acyclic. On the other hand, by the definition of the morphisms δ^i in (10.2.9), we see that N is just the standard cone on the strict morphism of complexes

$$T^{-1}(\rho) : T^{-1}(P) \rightarrow T^{-1}(M).$$

See Definition 3.9.1. Therefore ρ is a quasi-isomorphism. \square

Corollary 10.2.14. *If \mathbf{M} is an abelian category with enough projectives, then $\mathbf{C}^-(\mathbf{M})$ has enough K -projectives.*

Proof. Combine Theorem 10.2.7, Proposition 10.2.6 and Theorem 10.2.3. \square

For a graded object $N \in \mathbf{G}(\mathbf{M})$ we let

$$(10.2.15) \quad \sup(N) := \sup \{i \mid N^i \neq 0\} \subseteq \mathbb{Z} \cup \{\pm\infty\}.$$

Note that $\sup(N) = -\infty$ if and only if $N = 0$.

Corollary 10.2.16. *Let \mathbf{M} be an abelian category with enough projectives, and let $M \in \mathbf{C}(\mathbf{M})$ be a complex with bounded above cohomology. Then M has a K -projective resolution $P \rightarrow M$ with $\sup(P) = \sup(H(M))$.*

Proof. We may assume that $H(M)$ is not zero. Let $i := \sup(H(M)) \in \mathbb{Z}$, and take $N := \text{smt}^{\leq i}(M)$, the smart truncation from formula (7.3.6). Then $N \rightarrow M$ is a quasi-isomorphism and $\sup(N) = i$. An inspection of the proof of Theorem 10.2.7 shows that the resolution $P \rightarrow N$ can be made with $\sup(P) = i$. The composed quasi-isomorphism $P \rightarrow M$ has the desired properties. \square

Remark 10.2.17. Let \mathbf{M} be an abelian category, and let $\mathbf{N} \subseteq \mathbf{M}$ be a thick abelian subcategory. Assume that \mathbf{N} has enough \mathbf{M} -projectives, namely any object $N \in \mathbf{N}$ admits an epimorphism $P \twoheadrightarrow N$ where $P \in \mathbf{N}$ and it is projective as an object of \mathbf{M} . Suppose $M \in \mathbf{C}(\mathbf{M})$ is a complex satisfying these conditions: $H^i(M) \in \mathbf{N}$ for all i , and $H^i(M) = 0$ for $i \gg 0$. Then there is a quasi-isomorphism $P \rightarrow M$ in $\mathbf{C}_{\text{str}}(\mathbf{M})$, where the P^i are objects of \mathbf{N} that are projective as objects of \mathbf{M} , and $\sup(P) = \sup(H(M))$. See [RD, Lemma I.4.6(3)] for the reverse statement (with injectives). We will prove a slightly less general result for noetherian rings in Example 10.3.33.

10.3. K-Projective Resolutions in $\mathbf{C}(A)$. In this subsection A is a DG ring (without any vanishing assumption).

Recall that the translation $T^{-i}(A)$ is a DG A -module in which the element $t^{-i}(1)$ is in degree i . This element is a cocycle, and when we forget the differentials, the graded module $T^{-i}(A)^{\natural}$ is free over the graded ring A^{\natural} , with basis $t^{-i}(1)$. Therefore, for any DG A -module M there is a canonical isomorphism

$$(10.3.1) \quad \mathrm{Hom}_A(T^{-i}(A), M) \cong T^i(M)$$

in $\mathbf{C}(\mathbb{K})$, and canonical isomorphisms

$$(10.3.2) \quad \mathrm{Hom}_{\mathbf{C}_{\mathrm{str}}(A)}(T^{-i}(A), M) \cong Z^0(\mathrm{Hom}_A(T^{-i}(A), M)) \cong Z^i(M)$$

in $\mathbf{M}(\mathbb{K})$. (Actually, (10.3.1) is an isomorphism in $\mathbf{C}_{\mathrm{str}}(A)$, but this uses the DG A -bimodule structure of $T^{-i}(A)$.)

We begin with a definition that is very similar to Definition 10.2.2. Recall the notion of a filtration $F = \{F_j(P)\}_{j \geq -1}$ of a DG module P , and the associated subquotients $\mathrm{gr}_j^F(P)$ from formula (10.2.1).

Definition 10.3.3. Let P be an object of $\mathbf{C}(A)$.

- (1) We say that P is a *free DG A -module* if there is an isomorphism

$$P \cong \bigoplus_{s \in S} T^{-i_s}(A)$$

in $\mathbf{C}_{\mathrm{str}}(A)$, for some indexing set S and some collection of integers $\{i_s\}_{s \in S}$.

- (2) A *semi-free filtration* on P is a filtration $F = \{F_j(P)\}_{j \geq -1}$ of P in $\mathbf{C}_{\mathrm{str}}(A)$, such that:

- $F_{-1}(P) = 0$.
- Each $\mathrm{gr}_j^F(P)$ is a free DG A -module.
- $P = \bigcup_j F_j(P)$.

- (3) The DG module P is called *semi-free* if it admits some semi-free filtration.

Example 10.3.4. If A is a ring, then a free DG A -module P is a complex of free A -modules with zero differential. A semi-free DG A -module P is also a complex of free A -modules, but there is a differential on it, and there is a subtle condition on P imposed by the existence of a semi-free filtration. If the complex P happens to be bounded above, then it is automatically semi-free, with a filtration like the one in the proof of Proposition 10.2.6.

Exercise 10.3.5. Find a ring A , and a complex P of free A -modules, that is not semi-free. (Hint: Take the ring $A = \mathbb{K}[\epsilon]$ of dual numbers. Find a complex of free A -modules P that is acyclic but not null-homotopic. Now use Theorem 10.3.6 and Corollary 9.2.12 to get a contradiction.)

Theorem 10.3.6. Let P be an object of $\mathbf{C}(A)$. If P is semi-free, then it is K -projective.

Proof. It is similar to the proof of Theorem 10.2.3.

Step 1. We start by proving that if $P = T^{-i}(A)$, a translation of A , then P is K -projective. This is easy: given an acyclic $N \in \mathbf{C}(A)$, we have

$$\mathrm{Hom}_A(P, N) = \mathrm{Hom}_A(T^{-i}(A), N) \cong T^i(\mathrm{Hom}_A(A, N)) \cong T^i(N)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$, and this is acyclic.

Step 2. Now

$$P \cong \bigoplus_{s \in S} T^{-i_s}(A).$$

Then

$$\mathrm{Hom}_A(P, N) \cong \prod_{s \in S} \mathrm{Hom}_A(T^{-i_s}(A), N).$$

By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ is acyclic, we conclude that $\mathrm{Hom}_M(P, N)$ is acyclic.

Step 3. Fix a semi-free filtration $F = \{F_j(P)\}_{j \geq -1}$ of P . Here we prove that for every $j \geq -1$ the DG module $F_j(P)$ is K-projective. This is done by induction on $j \geq -1$. For $j = -1$ it is trivial. For $j \geq 0$ there is an exact sequence

$$(10.3.7) \quad 0 \rightarrow F_{j-1}(P) \rightarrow F_j(P) \rightarrow \mathrm{gr}_j^F(P) \rightarrow 0$$

in the abelian category $\mathbf{C}_{\mathrm{str}}(A)$. Because $\mathrm{gr}_j^F(P)$ is a free DG module, it is a projective object in the abelian category $\mathbf{G}^0(A^\natural)$ of graded modules over the graded ring A^\natural , gotten by forgetting the differential of A . Therefore the sequence (10.3.7) is split exact in $\mathbf{G}^0(A^\natural)$.

Let $N \in \mathbf{C}(A)$ be an acyclic DG module. Applying the functor $\mathrm{Hom}_A(-, N)$ to the sequence (10.3.7) we obtain a sequence

$$(10.3.8) \quad 0 \rightarrow \mathrm{Hom}_A(\mathrm{gr}_j^F(P), N) \rightarrow \mathrm{Hom}_A(F_j(P), N) \rightarrow \mathrm{Hom}_A(F_{j-1}(P), N) \rightarrow 0$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. If we forget differentials this is a sequence in $\mathbf{G}^0(\mathbb{K})$. Because (10.3.7) is split exact in $\mathbf{G}^0(A^\natural)$, it follows that (10.3.8) is split exact in $\mathbf{G}^0(\mathbb{K})$. Therefore (10.3.8) is exact in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the DG \mathbb{K} -module $\mathrm{Hom}_A(F_{j-1}(P), N)$ is acyclic. By step 2 the DG module $\mathrm{Hom}_A(\mathrm{gr}_j^F(P), N)$ is acyclic. The long exact cohomology sequence associated to (10.3.8) shows that the DG module $\mathrm{Hom}_A(F_j(P), N)$ is acyclic too.

Step 4. We keep the semi-free filtration $F = \{F_j(P)\}_{j \geq -1}$ from step 3. Take any acyclic $N \in \mathbf{C}(M)$. By Proposition 10.1.3 we know that

$$\mathrm{Hom}_A(P, N) \cong \varprojlim_j \mathrm{Hom}_A(F_j(P), N)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_A(F_j(P), N)$ are all acyclic. The exactness of the sequences (10.3.8) implies that the inverse system

$$\{\mathrm{Hom}_A(F_j(P), N)\}_{j \geq -1}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\mathrm{Hom}_A(P, N)$ is acyclic. \square

Here is a result similar to Theorem 10.2.7.

Theorem 10.3.9. *Let A be a DG ring. Any $M \in \mathbf{C}(A)$ admits a quasi-isomorphism $\rho : P \rightarrow M$ in $\mathbf{C}_{\mathrm{str}}(A)$ from a semi-free DG A -module P .*

Proof. Step 1. In this step we construct a free DG A -module $F_0(P)$ and a homomorphism $F_0(\rho) : F_0(P) \rightarrow M$. For any $i \in \mathbb{Z}$ the cohomology $H^i(M)$ is an $H^0(A)$ -module. Choose a collection of $H^0(A)$ -module generators of $H^i(M)$, indexed by a set S_0^i . There is a canonical surjection $Z^i(M) \twoheadrightarrow H^i(M)$, and we lift these

generators to a collection $\{m_s\}_{s \in S_0^i}$ of elements of $Z^i(M)$. Define the free DG A -module

$$(10.3.10) \quad Q_0^i := \bigoplus_{s \in S_0^i} T^{-i}(A).$$

The collection $\{m_s\}_{s \in S_0^i}$ induces a homomorphism

$$(10.3.11) \quad \phi_0^i : Q_0^i \rightarrow M$$

in $\mathbf{C}_{\text{str}}(A)$, as in formula (10.3.2). Define the free DG A -module

$$(10.3.12) \quad F_0(P) := \bigoplus_{i \in \mathbb{Z}} Q_0^i,$$

and let

$$(10.3.13) \quad F_0(\rho) : F_0(P) \rightarrow M, \quad F_0(\rho) := \sum_i \phi_0^i$$

be the resulting homomorphism in $\mathbf{C}_{\text{str}}(A)$. By construction we see that

$$(10.3.14) \quad H^i(F_0(\rho)) : H^i(F_0(P)) \rightarrow H^i(M)$$

is surjective for all i .

Step 2. In this step $j \geq 0$, and we are given the following: a DG A -module $F_j(P)$, a homomorphism $F_j(\rho) : F_j(P) \rightarrow M$ in $\mathbf{C}_{\text{str}}(A)$, and a filtration $\{F_{j'}(P)\}_{-1 \leq j' \leq j}$ of $F_j(P)$. These satisfy the following conditions: for all i and all $0 \leq j' \leq j$ the homomorphisms

$$(10.3.15) \quad H^i(F_j(\rho)) : H^i(F_{j'}(P)) \rightarrow H^i(M)$$

are surjective; $F_{-1}(P) = 0$; and the DG A -modules $\text{gr}_{j'}^F(P)$ are free for all $0 \leq j' \leq j$.

For any $i \in \mathbb{Z}$ let K_j^i be the kernel of $H^i(F_j(\rho))$. So there is a short exact sequence

$$(10.3.16) \quad 0 \rightarrow K_j^i \rightarrow H^i(F_j(P)) \xrightarrow{H^i(F_j(\rho))} H^i(M) \rightarrow 0$$

in $\mathbf{M}(H^0(A))$. Choose a collection of $H^0(A)$ -module generators of K_j^i , indexed by a set S_{j+1}^i . Using the canonical surjection $Z^i(F_j(P)) \twoheadrightarrow H^i(F_j(P))$, lift these generators to a collection $\{p_s\}_{s \in S_{j+1}^i}$ of elements of the module of cocycles $Z^i(F_j(P))$. Define the free DG A -module

$$(10.3.17) \quad Q_{j+1}^i := \bigoplus_{s \in S_{j+1}^i} T^{-i}(A).$$

The collection of cocycles $\{p_s\}_{s \in S_{j+1}^i}$ induces a homomorphism

$$(10.3.18) \quad \phi_{j+1}^i : Q_{j+1}^i \rightarrow F_j(P)$$

in $\mathbf{C}_{\text{str}}(A)$. Next define the free DG A -module

$$(10.3.19) \quad Q_{j+1} := \bigoplus_{i \in \mathbb{Z}} Q_{j+1}^i$$

and the homomorphism

$$(10.3.20) \quad \phi_{j+1} : Q_{j+1} \rightarrow F_j(P), \quad \phi_{j+1} := \sum_i \phi_{j+1}^i$$

in $\mathbf{C}_{\text{str}}(A)$.

Now let us define the DG A -module $F_{j+1}(P)$ by attaching Q_{j+1} to $F_j(P)$ along ϕ_{j+1} . Namely, as a graded module we let

$$(10.3.21) \quad F_{j+1}(P)^{\natural} := F_j(P)^{\natural} \oplus T(Q_{j+1})^{\natural},$$

and the differential is

$$d_{F_{j+1}(P)} := d_{F_j(P)} + d_{T(Q_{j+1})} + \phi_{j+1} \circ t^{-1}.$$

In other words, $F_{j+1}(P)$ is the standard cone on the strict homomorphism ϕ_{j+1} ; see Definition 3.9.1. We note that the basis of the free DG module Q_{j+1}^i sits inside $F_{j+1}(P)^{i-1}$.

By construction, $F_j(P)$ is a DG submodule of $F_{j+1}(P)$. Let us denote the inclusion by

$$\mu_j : F_j(P) \hookrightarrow F_{j+1}(P).$$

Because the cocycles in $F_j(P)$ representing K_j^i become coboundaries in $F_{j+1}(P)$, it follows that for any i we have

$$(10.3.22) \quad K_j^i \subseteq \text{Ker}(H^i(\mu_j) : H^i(F_j(P)) \rightarrow H^i(F_{j+1}(P))).$$

Step 3. In this step we construct the homomorphism $F_{j+1}(\rho)$, continuing from where we left off in step 2. Consider the element $p_s \in Z^i(F_j(P))$ for some index $s \in S_{j+1}^i$. Because the cohomology class of p_s is in K_j^i , the element $F_j(\rho)(p_s) \in M^i$ is a coboundary. Therefore we can find an element $m_s \in M^{i-1}$ such that $F_j(\rho)(p_s) = d_M(m_s)$. From (10.3.19) we see that the collection of elements $\{m_s\}_{s \in \coprod_i S_{j+1}^i}$ induces a strict homomorphism of DG modules

$$\rho'_{j+1} : T(Q_{j+1}) \rightarrow M.$$

Define the homomorphism

$$F_{j+1}(\rho) : F_{j+1}(P) \rightarrow M$$

to be

$$F_{j+1}(\rho) := F_j(\rho) + \rho'_{j+1}$$

using the direct sum decomposition (10.3.21). It is easy to check that this is a strict homomorphism of DG modules.

Step 4. After going through steps 2 and 3 inductively, we now have a direct system $\{F_j(P)\}_{j \geq -1}$ in $\mathbf{C}_{\text{str}}(A)$, and a direct system of homomorphisms $F_j(\rho) : F_j(P) \rightarrow M$. Define the DG A -module

$$P := \lim_{j \rightarrow} F_j(P)$$

and the homomorphism

$$\rho := \lim_{j \rightarrow} F_j(\rho) : P \rightarrow M$$

in $\mathbf{C}_{\text{str}}(A)$. The DG module P has on it the filtration $\{F_j(P)\}$, and it is a semi-free filtration. Indeed, there are isomorphisms $\text{gr}_0^F(P) \cong \bigoplus_{i \in \mathbb{Z}} Q_0^i$ and $\text{gr}_{j+1}^F(P) \cong T(Q_{j+1})$ for $j \geq 0$.

It remains to prove that ρ is a quasi-isomorphism. We know that the homomorphisms $H^i(F_j(\rho))$ are surjective for all i and all $j \geq 0$. Define

$$L_j^i := \text{Im}(H^i(\mu_j) : H^i(F_j(P)) \rightarrow H^i(F_{j+1}(P))) \subseteq H^i(F_{j+1}(P)).$$

We get a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_j^i & \xrightarrow{\text{inc}} & H^i(F_j(P)) & \xrightarrow{H^i(F_j(\rho))} & H^i(M) \longrightarrow 0 \\
 & & & & \downarrow \alpha_j^i & \nearrow \beta_j^i & \\
 & & & & L_j^i & & \\
 & & & & \downarrow \text{inc} & \nearrow H^i(F_{j+1}(\rho)) & \\
 & & & & H^i(F_{j+1}(P)) & &
 \end{array}$$

$H^i(\mu_j)$ is indicated by a curved arrow from K_j^i to $H^i(F_{j+1}(P))$.

in $\mathbf{M}(\mathbb{K})$. The top row is an exact sequence (it is (10.3.16)). Because α_j^i is surjective, there is equality

$$\text{Ker}(\beta_j^i) = \alpha_j^i(\text{Ker}(H^i(F_j(\rho)))) = \alpha_j^i(K_j^i).$$

But by formula (10.3.22) we know that $\alpha_j^i(K_j^i) = 0$. The conclusion is that

$$(10.3.23) \quad \beta_j^i : L_j^i \rightarrow H^i(M)$$

is an isomorphism. Hence, for every i the direct system $\{L_j^i\}_{j \geq 0}$ has a limit, and the homomorphism

$$(10.3.24) \quad \lim_{j \rightarrow} L_j^i \rightarrow H^i(M)$$

is bijective. Now the direct systems $\{L_j^i\}_{j \geq 0}$ and $\{H^i(F_j(P))\}_{j \geq 0}$ are sandwiched; so by Proposition 10.1.1(1) we know that the second direct system also has a limit, and the canonical homomorphism

$$(10.3.25) \quad \lim_{j \rightarrow} H^i(F_j(P)) \rightarrow \lim_{j \rightarrow} L_j^i$$

is bijective. Finally, according to Proposition 10.1.7 we know that the canonical homomorphism

$$(10.3.26) \quad \lim_{j \rightarrow} H^i(F_j(P)) \rightarrow H^i(P)$$

is bijective. The combination of the bijections (10.3.24), (10.3.25) and (10.3.26) implies that

$$H^i(\rho) : H^i(P) \rightarrow H^i(M)$$

is bijective. □

Corollary 10.3.27. *Let A be any DG ring. The category $\mathbf{C}(A)$ has enough K -projectives.*

Proof. Combine Theorems 10.3.6 and 10.3.9. □

The concept of nonpositive DG ring was introduced in Definition 7.3.11.

Corollary 10.3.28. *Assume A is a nonpositive DG ring. For any $M \in \mathbf{C}(A)$ there is a K -projective resolution $P \rightarrow M$ with $\sup(P) = \sup(H(M))$.*

Proof. If $H(M)$ is unbounded above or zero, the assertion is trivial. So we may assume that $i_1 := \sup(H(M))$ is an integer. In steps 1 and 2 of the proof of Theorem 10.3.9 we choose the indexing sets S_j^i to be empty whenever this is possible. Namely $S_0^i = \emptyset$ when $H^i(M) = 0$, and $S_{j+1}^i = \emptyset$ when $K_j^i = 0$. We claim that with these choices, the inductive construction will satisfy the following extra condition: the homomorphisms

$$(10.3.29) \quad H^i(F_j(\rho)) : H^i(F_j(P)) \rightarrow H^i(M)$$

are bijective for all $i \geq i_1 + 1 - j$. This in turn implies that $K_j^i = 0$ for all $i \geq i_1 + 1 - j$. We see that $K_j^i = 0$ and $H^i(M) = 0$ for all $i \geq i_1 + 1$. Since A is nonpositive, this says that $\sup(F_j(P)) \leq i_1$. Therefore in the limit we get $\sup(P) \leq i_1$.

Let us prove the claim, by induction on $j \geq 0$. For $j = 0$ this is trivial, because both modules in (10.3.29) vanish for $i \geq 1$. Now assume that $j \geq 0$ and the claim holds. So $K_j^i = 0$ for all $i \geq i_1 + 1 - j$. Then, by formula (10.3.21), the DG module $F_{j+1}(P)$ coincides with its submodule $F_j(P)$ in degrees $\geq i_1 - j$. This implies that these DG modules have the same cohomologies in degrees $\geq i_1 - j + 1$, and the same cocycles in degree $i_1 - j$. Thus the homomorphisms $H^i(F_{j+1}(\rho))$ remain bijective for $i \geq i_1 - j + 1$. These homomorphisms are surjective for all i . But in $F_{j+1}(P)$ there are new coboundaries in degree $i_1 - j$, those coming from $Q_{j+1}^{i_1-j-1}$. These cocycles cause the homomorphism $H^{i_1-j}(F_{j+1}(\rho))$ to be injective. So the inductive step is completed. \square

Definition 10.3.30. Let A be a nonpositive DG ring. A DG A -module P is called *pseudo-finite semi-free* if it admits a semi-free filtration $F = \{F_j(P)\}_{j \geq -1}$ satisfying this extra condition: there are $i_1 \in \mathbb{Z}$ and $r_j \in \mathbb{N}$ such that

$$\mathrm{gr}_j^F(P) \cong T^{-i_1+j}(A)^{\oplus r_j}$$

in $\mathbf{C}_{\mathrm{str}}(A)$ for all j .

Exercise 10.3.31. Let A be a nonpositive DG ring and let P be a DG A -module. Prove that the following two conditions are equivalent.

- (i) P is pseudo-finite semi-free.
- (ii) There are numbers $i_1 \in \mathbb{Z}$ and $r_j \in \mathbb{N}$, and an isomorphism

$$P^{\natural} \cong \bigoplus_{j \geq 0} T^{-i_1+j}(A^{\natural})^{\oplus r_j}$$

in $\mathbf{G}^0(A^{\natural})$.

In case A is a ring (i.e. $A^i = 0$ for all $i \neq 0$), prove that these conditions are equivalent to:

- (iii) P is a bounded above complex of finitely generated free A -modules.

Corollary 10.3.32. Assume that A is a nonpositive DG ring, and the ring $H^0(A)$ is left noetherian. Let M be a DG A -module satisfying these conditions: each $H^i(M)$ is a finitely generated $H^0(A)$ -module, and $H^i(M) = 0$ for $i \gg 0$. Then there is a quasi-isomorphism $P \rightarrow M$ in $\mathbf{C}_{\mathrm{str}}(A)$ from a pseudo-finite semi-free DG A -module P with $\sup(P) = \sup(H(M))$.

Proof. Like in the proof of Corollary 10.3.28, the key to the proof is to economize. Besides the choice of empty indexing sets S_j^i that we imposed there, here we

choose all these sets to be finite. This is possible, since the $H^0(A)$ -modules $H^i(M)$, $H^i(F_j(P))$ and K_j^i will all be finitely generated. \square

Example 10.3.33. A special yet very important case of Corollary 10.3.32 is this: A is a left noetherian ring, and M is a complex of A -modules with bounded above cohomology, such that each $H^i(M)$ is a finitely generated A -module. Then M has a resolution $P \rightarrow M$, where P is a complex of finitely generated free A -modules, and $\sup(P) = \sup(H(M))$.

10.4. K-Injective Resolutions in $\mathbf{C}^+(\mathbf{M})$. In this subsection \mathbf{M} is an abelian category, and $\mathbf{C}(\mathbf{M})$ is the category of complexes in \mathbf{M} .

In subsection 1.3 we discussed quotients in categories. A *cofiltration* of a complex $I \in \mathbf{C}(\mathbf{M})$ is an inverse system $G = \{G_q(I)\}_{q \geq -1}$ of quotients of I in $\mathbf{C}_{\text{str}}(\mathbf{M})$. We say that $I = \lim_{\leftarrow q} G_q(I)$ if this inverse limit exists in $\mathbf{C}_{\text{str}}(\mathbf{M})$, and the canonical morphism $I \rightarrow \lim_{\leftarrow q} G_q(I)$ is an isomorphism. The cofiltration G gives rise to the subquotients

$$(10.4.1) \quad \text{gr}_q^G(I) := \text{Ker}(G_q(I) \rightarrow G_{q-1}(I)) \in \mathbf{C}(\mathbf{M}).$$

Definition 10.4.2. Let I be a complex in $\mathbf{C}(\mathbf{M})$.

- (1) A *semi-injective cofiltration* on I is a cofiltration $G = \{G_q(I)\}_{q \geq -1}$ in $\mathbf{C}_{\text{str}}(\mathbf{M})$ such that:
 - $G_{-1}(I) = 0$.
 - Each $\text{gr}_q^G(I)$ is a complex of injective objects of \mathbf{M} with zero differential.
 - $I = \lim_{\leftarrow q} G_q(I)$.
- (2) The complex I is called a *semi-injective complex* if it admits some semi-injective cofiltration.

Theorem 10.4.3. Let \mathbf{M} be an abelian category, and let I be a semi-injective complex in $\mathbf{C}(\mathbf{M})$. Then I is K-injective.

Proof. The proof is very similar to that of Theorem 10.2.3.

Step 1. We start by proving that if $I = T^p(J)$, the translation of an injective object $J \in \mathbf{M}$, then I is K-injective. This is easy: given an acyclic complex $N \in \mathbf{C}(\mathbf{M})$, we have

$$\text{Hom}_{\mathbf{M}}(N, I) = \text{Hom}_{\mathbf{M}}(N, T^p(J)) \cong T^p(\text{Hom}_{\mathbf{M}}(N, J))$$

in $\mathbf{C}_{\text{str}}(\mathbb{K})$. But $\text{Hom}_{\mathbf{M}}(-, J)$ is an exact functor $\mathbf{M} \rightarrow \mathbf{M}(\mathbb{K})$, so $\text{Hom}_{\mathbf{M}}(N, J)$ is an acyclic complex.

Step 2. Now I is a complex of injective objects of \mathbf{M} with zero differential. This means that

$$I \cong \prod_{p \in \mathbb{Z}} T^p(J_p)$$

in $\mathbf{C}_{\text{str}}(\mathbf{M})$, where each J_p is an injective object in \mathbf{M} . But then

$$\text{Hom}_{\mathbf{M}}(N, I) \cong \prod_{p \in \mathbb{Z}} \text{Hom}_{\mathbf{M}}(N, T^p(J_p)).$$

This is an easy case of Proposition 10.1.3(2). By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}_{\text{str}}(\mathbb{K})$ is acyclic (itself an easy case of the Mittag-Leffler argument), we conclude that $\text{Hom}_{\mathbf{M}}(N, I)$ is acyclic.

Step 3. Fix a semi-injective cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I . Here we prove that for every q the complex $G_q(I)$ is K-injective. This is done by induction on q . For $q = -1$ it is trivial. For $q \geq 0$ there is an exact sequence of complexes

$$(10.4.4) \quad 0 \rightarrow \operatorname{gr}_q^G(I) \rightarrow G_q(I) \rightarrow G_{q-1}(I) \rightarrow 0$$

in $\mathbf{C}_{\text{str}}(\mathbf{M})$. In each degree $p \in \mathbb{Z}$ the exact sequence

$$0 \rightarrow \operatorname{gr}_q^G(I)^p \rightarrow G_q(I)^p \rightarrow G_{q-1}(I)^p \rightarrow 0$$

in \mathbf{M} splits, because $\operatorname{gr}_q^G(I)^p$ is an injective object. Thus the exact sequence (10.4.4) is split in the category $\mathbf{G}^0(\mathbf{M})$ of graded objects in \mathbf{M} .

Let $N \in \mathbf{C}(\mathbf{M})$ be an acyclic complex. Applying the functor $\operatorname{Hom}_{\mathbf{M}}(N, -)$ to the sequence of complexes (10.4.4) we obtain a sequence

$$(10.4.5) \quad 0 \rightarrow \operatorname{Hom}_{\mathbf{M}}(N, \operatorname{gr}_q^G(I)) \rightarrow \operatorname{Hom}_{\mathbf{M}}(N, G_q(I)) \rightarrow \operatorname{Hom}_{\mathbf{M}}(N, G_{q-1}(I)) \rightarrow 0$$

in $\mathbf{C}_{\text{str}}(\mathbb{K})$. Because (10.4.4) is split exact in $\mathbf{G}^0(\mathbf{M})$, the sequence (10.4.5) is split exact in $\mathbf{G}^0(\mathbb{K})$. Therefore (10.4.5) is exact in $\mathbf{C}_{\text{str}}(\mathbb{K})$.

By the induction hypothesis the complex $\operatorname{Hom}_{\mathbf{M}}(N, G_{q-1}(I))$ is acyclic. By step 2 the complex $\operatorname{Hom}_{\mathbf{M}}(N, \operatorname{gr}_q^G(I))$ is acyclic. The long exact cohomology sequence associated to (10.4.5) shows that the complex $\operatorname{Hom}_{\mathbf{M}}(N, G_q(I))$ is acyclic too.

Step 4. We keep the semi-injective cofiltration $G = \{G_q(I)\}_{q \geq -1}$ from step 3. Take any acyclic complex $N \in \mathbf{C}(\mathbf{M})$. By Proposition 10.1.3 we know that

$$\operatorname{Hom}_{\mathbf{M}}(N, I) \cong \varprojlim_q \operatorname{Hom}_{\mathbf{M}}(N, G_q(I))$$

in $\mathbf{C}_{\text{str}}(\mathbb{K})$. According to step 3 the complexes $\operatorname{Hom}_{\mathbf{M}}(N, G_q(I))$ are all acyclic. The exactness of the sequences (10.4.5) implies that the inverse system

$$\{\operatorname{Hom}_{\mathbf{M}}(N, G_q(I))\}_{q \geq -1}$$

in $\mathbf{C}_{\text{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\operatorname{Hom}_{\mathbf{M}}(N, I)$ is acyclic. \square

Proposition 10.4.6. *Let \mathbf{M} be an abelian category. If I is a bounded below complex of injectives, then I is a semi-injective complex.*

Proof. We can assume that $I \neq 0$. Let p_0 be an integer such that $I^p = 0$ for all $p < p_0$. For $q \geq -1$ let $F_q(I)$ be the subcomplex of I defined by $F_q(I)^p := I^p$ if $p \geq p_0 + q + 1$, and $F_q(I)^p := 0$ otherwise. Then let $G_q(I) := I/F_q(I)$. The cofiltration $G = \{G_q(I)\}_{q \geq -1}$ is semi-injective. \square

The next theorem is [RD, Lemma 4.6(1)]. See also [KaSc1, Proposition 1.7.7(i)].

Theorem 10.4.7. *Let \mathbf{M} be an abelian category with enough injectives. Any complex $M \in \mathbf{C}^+(\mathbf{M})$ admits a quasi-isomorphism $\rho : M \rightarrow I$ in $\mathbf{C}_{\text{str}}^+(\mathbf{M})$ into a bounded below complex of injectives I .*

Proof. The proof is the same as that of Theorem 10.2.7, except for a mechanical reversal of arrows. This is because of the symmetry of the axioms of abelian categories, that exchanges projective and injective objects. \square

Exercise 10.4.8. Try to write an explicit proof of Theorem 10.4.7. You will see that this requires a lot of patience. It is a bit like trying to write mirror-flipped text (i.e. text that looks normal when reflected in a mirror).

Remark 10.4.9. One must be careful about using the symmetry of the axioms of abelian categories. It is valid for finitary constructions (as in the proof of Theorem 10.4.7); but it might break down for constructions where limits are involved, such as in the proof of Theorem 10.4.3.

Corollary 10.4.10. *If \mathbf{M} is an abelian category with enough injectives, then $\mathbf{C}^+(\mathbf{M})$ has enough K -injectives.*

Proof. Combine Theorems 10.4.7, Proposition 10.4.6 and Theorem 10.4.3. \square

For a graded object $N \in \mathbf{G}(\mathbf{M})$ we let

$$(10.4.11) \quad \inf(N) := \inf \{i \mid N^i \neq 0\} \subseteq \mathbb{Z} \cup \{\pm\infty\}.$$

Note that $\inf(N) = \infty$ if and only if $N = 0$.

Corollary 10.4.12. *Let \mathbf{M} be an abelian category with enough injectives, and let $M \in \mathbf{C}(\mathbf{M})$ be a complex with bounded below cohomology. Then M has a K -injective resolution $M \rightarrow I$ with $\inf(I) = \inf(H(M))$.*

Proof. We may assume that $H(M)$ is nonzero. Let $p := \inf(H(M)) \in \mathbb{Z}$, and let $N := \text{smt}^{\geq p}(M)$, the smart truncation from formula 7.3.7. So $M \rightarrow N$ is a quasi-isomorphism. By proof of Theorem 10.2.7 (with sides flipped) we see that there is a K -injective resolution $N \rightarrow I$ with $\inf(I) = \inf(H(M))$. The composed quasi-isomorphism $M \rightarrow I$ is what we are looking for. \square

Remark 10.4.13. Let \mathbf{M} be an abelian category, and let $\mathbf{N} \subseteq \mathbf{M}$ be a thick abelian subcategory. Assume that \mathbf{N} has enough \mathbf{M} -injectives, namely any object $N \in \mathbf{N}$ admits a monomorphism $N \hookrightarrow I$ where $I \in \mathbf{N}$ and it is injective as an object of \mathbf{M} . Suppose $M \in \mathbf{C}(\mathbf{M})$ is a complex satisfying these conditions: $H^i(M) \in \mathbf{N}$ for all i , and $H^i(M) = 0$ for $i \ll 0$. Then there is a quasi-isomorphism $M \rightarrow I$ in $\mathbf{C}_{\text{str}}(\mathbf{M})$, where the I^p are objects of \mathbf{N} that are injective as objects of \mathbf{M} , and $\inf(I) = \inf(H(M))$. See [RD, Lemma I.4.6(3)] for a very sketchy proof.

An important example is this: (X, \mathcal{O}_X) is a noetherian scheme, $\mathbf{M} = \text{Mod } \mathcal{O}_X$ and $\mathbf{N} = \text{QCoh } \mathcal{O}_X$. According to [RD, Proposition II.7.6] the category \mathbf{N} has enough \mathbf{M} -injectives.

10.5. K -Injective Resolutions in $\mathbf{C}(A)$. Recall that we are working over a nonzero commutative base ring \mathbb{K} , and A is a central DG \mathbb{K} -ring.

An *injective cogenerator* of the abelian category $\mathbf{M}(\mathbb{K}) = \text{Mod } \mathbb{K}$ is an injective \mathbb{K} -module \mathbb{K}^* with this property: if M is a nonzero \mathbb{K} -module, then $\text{Hom}_{\mathbb{K}}(M, \mathbb{K}^*)$ is nonzero. These always exist. Here are a few examples.

Example 10.5.1. For any nonzero ring \mathbb{K} there is a canonical choice for an injective cogenerator:

$$\mathbb{K}^* := \text{Hom}_{\mathbb{Z}}(\mathbb{K}, \mathbb{Q}/\mathbb{Z}).$$

See proof of Theorem 2.6.10. Usually this a very big module!

Example 10.5.2. Assume \mathbb{K} is a complete noetherian local ring, with maximal ideal \mathfrak{m} and residue field $\mathbb{k} = \mathbb{K}/\mathfrak{m}$. In this case we would prefer to take the smallest possible injective cogenerator \mathbb{K}^* , and this is the injective hull of \mathbb{k} as a \mathbb{K} -module.

Here are some special cases. If \mathbb{K} is a field, then $\mathbb{K}^* = \mathbb{K} = \mathbb{k}$. If $\mathbb{K} = \widehat{\mathbb{Z}}_p$, the ring of p -adic integers, then $\mathbb{k} = \mathbb{F}_p$, and $\mathbb{K}^* \cong \widehat{\mathbb{Q}}_p/\widehat{\mathbb{Z}}_p$, which is the p -primary part of \mathbb{Q}/\mathbb{Z} . If \mathbb{K} contains some field, then there exists a ring homomorphism $\mathbb{k} \rightarrow \mathbb{K}$

that lifts the canonical surjection $\mathbb{K} \rightarrow \mathbb{k}$. After choosing such a lifting, there is an isomorphism of \mathbb{K} -modules

$$\mathbb{K}^* \cong \mathrm{Hom}_{\mathbb{k}}^{\mathrm{cont}}(\mathbb{K}, \mathbb{k}),$$

where continuity is for the \mathfrak{m} -adic topology on \mathbb{K} and the discrete topology on \mathbb{k} .

In this subsection we fix an injective cogenerator \mathbb{K}^* of $\mathbf{M}(\mathbb{K})$. For any $p \in \mathbb{Z}$ there is the DG \mathbb{K} -module $T^{-p}(\mathbb{K}^*)$, which is concentrated in degree p , and has the trivial differential.

Definition 10.5.3. A DG \mathbb{K} -module W is called *cofree* if

$$W \cong \prod_{s \in S} T^{-p_s}(\mathbb{K}^*)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$, for some indexing set S and some collection of integers $\{p_s\}_{s \in S}$.

The differential of a cofree DG \mathbb{K} -module W is trivial. It is not hard to see that W is a \mathbb{K} -injective DG \mathbb{K} -module. When we view W as a graded \mathbb{K} -module, i.e. as an object of the abelian category $\mathbf{G}^0(\mathbb{K})$, it is injective.

A few more words on the structure of cofree DG \mathbb{K} -modules. Let's partition the set S as follows: $S = \coprod_{p \in \mathbb{Z}} S^p$, where $S^p := \{s \in S \mid p_s = p\}$. Then $W^p = \prod_{s \in S^p} \mathbb{K}^*$ as \mathbb{K} -modules.

Remark 10.5.4. It will be convenient to blur the distinction between DG modules with zero differentials and graded modules. Specifically, let N be a DG module such that $d_N = 0$. We are going to identify N with the graded modules N^{\natural} and $H(N)$. Typical examples are these: a cofree DG \mathbb{K} -module W , and the DG modules $Z(M)$ and $B(M)$ arising from any DG module M .

Lemma 10.5.5. *Let M be a DG \mathbb{K} -module, let W be a cofree DG \mathbb{K} -module, and let $\xi : M \rightarrow W$ be a homomorphism in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. If $H(\xi) : H(M) \rightarrow W$ is the zero homomorphism, then ξ is a coboundary in the DG module $\mathrm{Hom}_{\mathbb{K}}(M, W)$.*

Proof. Because W has zero differential, the homomorphism $H(\xi)$ is zero iff $\xi|_{Z(M)} : Z(M) \rightarrow W$ is zero. Consider the exact sequence

$$0 \rightarrow Z(M) \rightarrow M^{\natural} \xrightarrow{t \circ d_M} T(B(M)) \rightarrow 0$$

in $\mathbf{G}^0(\mathbb{K})$. Applying $\mathrm{Hom}_{\mathbb{K}}(-, W)$, and taking only the degree 0 part, we obtain the exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathbb{K}}(B(M), W)^{-1} \xrightarrow{\mathrm{Hom}(d, \mathrm{id})} \mathrm{Hom}_{\mathbb{K}}(M, W)^0 \rightarrow \mathrm{Hom}_{\mathbb{K}}(Z(M), W)^0 \rightarrow 0.$$

We are using the fact that W is injective in $\mathbf{G}^0(\mathbb{K})$. The homomorphism ξ lives in the middle term, and it goes to zero in the right term; hence it comes from some ζ in the left term. Thus $\xi = \zeta \circ d$ for a degree -1 homomorphism $\zeta : B(M) \rightarrow W$. Again using the fact that W is injective in $\mathbf{G}^0(\mathbb{K})$, and considering the embedding $B(M) \hookrightarrow M^{\natural}$, we see that ζ extends to a degree -1 homomorphism $\zeta : M^{\natural} \rightarrow W$. \square

Exercise 10.5.6. In the situation of Lemma 10.5.5, prove that there is a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{K}}(H(M), W) \cong H(\mathrm{Hom}_{\mathbb{K}}(M, W))$$

in $\mathbf{G}^0(\mathbb{K})$. (Hint: look at the proof of [PSY, Corollary 2.12].)

Lemma 10.5.7. *Let M be a DG \mathbb{K} -module with zero differential. There is an injective homomorphism $\chi : M \rightarrow W$ into some cofree DG \mathbb{K} -module W .*

Proof. It is enough to prove that for any nonzero element $m \in M^p$ there is a homomorphism $\chi_m : M^p \rightarrow \mathbb{K}^*$ such that $\chi_m(m) \neq 0$. This is a direct consequence of the fact that \mathbb{K}^* is an injective cogenerator; see the proof of Theorem 2.6.10 for details. \square

Definition 10.5.8. Let W be a cofree DG \mathbb{K} -module. The *cofree DG A -module coinduced from W* is the DG A -module

$$I_W := \text{Hom}_{\mathbb{K}}(A, W).$$

There is a homomorphism

$$\theta_W : I_W \rightarrow W, \quad \theta(\chi) := \chi(1)$$

in $\mathbf{C}_{\text{str}}(\mathbb{K})$. It is called the *trace*.

Definition 10.5.9. A DG A -module I is called *cofree* if there is an isomorphism $I \cong I_W$ in $\mathbf{C}_{\text{str}}(A)$ for some cofree DG \mathbb{K} -module W .

A special cofree DG A -module is $A^* := \text{Hom}_{\mathbb{K}}(A, \mathbb{K}^*)$. Any other cofree DG module I is built from A^* , in the sense that there is an isomorphism

$$I \cong \prod_{s \in S} T^{-p_s}(A^*)$$

in $\mathbf{C}_{\text{str}}(A)$, using the notation of Definition 10.5.3.

Lemma 10.5.10 (Adjunction). *Let W be a cofree DG \mathbb{K} -module, and let M be a DG A -module. The homomorphism*

$$\text{Hom}(\text{id}_M, \theta_W) : \text{Hom}_A(M, I_W) \rightarrow \text{Hom}_{\mathbb{K}}(M, W)$$

in $\mathbf{C}_{\text{str}}(\mathbb{K})$ is an isomorphism.

Proof. Given $\chi \in \text{Hom}_{\mathbb{K}}(M, W)^p$, let $\phi : M \rightarrow I_W$ be the function

$$\phi(m)(a) := (-1)^{ql} \cdot \chi(a \cdot m) \in W$$

for $m \in M^q$ and $a \in A^l$. Then $\phi \in \text{Hom}_A(M, I_W)^p$, and

$$\text{Hom}(\text{id}_M, \theta)(\phi) = \theta \circ \phi = \chi.$$

We see that $\chi \mapsto \phi$ is an inverse of $\text{Hom}(\text{id}_M, \theta)$. \square

Recall that $\mathbf{G}^0(A^\natural)$ is the abelian category whose objects are the graded A^\natural -modules, and the morphisms are the A -linear homomorphisms of degree 0. The forgetful functor

$$\mathbf{C}_{\text{str}}(A) \rightarrow \mathbf{G}^0(A^\natural), \quad M \mapsto M^\natural,$$

is faithful.

Lemma 10.5.11. *Let I be a cofree DG A -module. Then I^\natural is an injective object of $\mathbf{G}^0(A^\natural)$.*

Proof. We can assume that $I = I_W$ for some cofree DG \mathbb{K} -module W . For any $M \in \mathbf{G}^0(A^\natural)$ there are isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{G}^0(A^\natural)}(M, I_W^\natural) &= \mathrm{Hom}_A(M, I_W)^0 \\ &\cong^{\heartsuit} \mathrm{Hom}_{\mathbb{K}}(M, W)^0 = \prod_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M^p, W^p). \end{aligned}$$

The isomorphism \cong^{\heartsuit} is by Lemma 10.5.10. For every p the functor $\mathbf{G}^0(A^\natural) \rightarrow \mathbf{M}(\mathbb{K})$, $M \mapsto M^p$, is exact. Because each W^p is an injective object of $\mathbf{M}(\mathbb{K})$, the functor $\mathrm{Hom}_{\mathbb{K}}(-, W^p)$ is exact. And the product of exact functors into $\mathbf{M}(\mathbb{K})$ is exact. We conclude that the functor $\mathrm{Hom}_{\mathbf{G}^0(A^\natural)}(-, I_W^\natural)$ is exact. \square

The next definition is dual to Definition 10.3.3.

Definition 10.5.12. Let I be an object of $\mathbf{C}(A)$.

- (1) A *semi-cofree cofiltration* on I is a cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I in $\mathbf{C}_{\mathrm{str}}(A)$ such that:
 - $G_{-1}(I) = 0$.
 - Each $\mathrm{gr}_q^G(I)$ is a cofree DG A -module.
 - $I = \lim_{\leftarrow q} G_q(I)$.
- (2) The DG A -module I is called a *semi-cofree* if it admits a semi-cofree cofiltration.

Theorem 10.5.13. Let I be an object of $\mathbf{C}(A)$. If I is semi-cofree, then it is K -injective.

Proof. The proof is very similar to those of Theorems 10.2.3 and 10.3.6. But because the arguments involve limits, we shall give the full proof.

Step 1. Suppose I is cofree; say $I \cong \prod_{s \in S} T^{-p_s}(A^*)$. The adjunction formula (Lemma 10.5.10) implies that for any DG A -module N there is an isomorphism

$$\mathrm{Hom}_A(N, I) \cong \prod_{s \in S} \mathrm{Hom}_{\mathbb{K}}(T^{p_s}(N), \mathbb{K}^*)$$

of graded \mathbb{K} -modules. It follows that if N is acyclic, then so is $\mathrm{Hom}_A(N, I)$.

Step 2. Fix a semi-cofree cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I . Here we prove that for every $q \geq -1$ the DG module $G_q(I)$ is K -injective. This is done by induction on $q \geq -1$. For $q = -1$ it is trivial. For $q \geq 0$ there is an exact sequence

$$(10.5.14) \quad 0 \rightarrow \mathrm{gr}_q^F(I) \rightarrow G_q(I) \rightarrow G_{q-1}(I) \rightarrow 0$$

in the category $\mathbf{C}_{\mathrm{str}}(A)$. Because $\mathrm{gr}_q^G(I)$ is a cofree DG A -module, it is an injective object in the abelian category $\mathbf{G}^0(A^\natural)$; see Lemma 10.5.11. Therefore the sequence (10.5.14) is split exact in $\mathbf{G}^0(A^\natural)$.

Let $N \in \mathbf{C}(A)$ be an acyclic DG module. Applying the functor $\mathrm{Hom}_A(N, -)$ to the sequence (10.5.14) we obtain a sequence

$$(10.5.15) \quad 0 \rightarrow \mathrm{Hom}_A(N, \mathrm{gr}_q^G(I)) \rightarrow \mathrm{Hom}_A(N, G_q(I)) \rightarrow \mathrm{Hom}_A(N, G_{q-1}(I)) \rightarrow 0$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. If we forget differentials this is a sequence in $\mathbf{G}^0(\mathbb{K})$. Because (10.5.14) is split exact in $\mathbf{G}^0(A^\natural)$, it follows that (10.5.15) is split exact in $\mathbf{G}^0(\mathbb{K})$. Therefore (10.5.15) is exact in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the DG \mathbb{K} -module $\mathrm{Hom}_A(N, G_{q-1}(I))$ is acyclic. By step 1 the DG \mathbb{K} -module $\mathrm{Hom}_A(N, \mathrm{gr}_q^G(I))$ is acyclic. The long exact cohomology sequence associated to (10.5.15) shows that the DG \mathbb{K} -module $\mathrm{Hom}_A(N, G_q(I))$ is acyclic too.

Step 3. We keep the semi-cofree cofiltration $G = \{G_q(I)\}_{q \geq -1}$ from step 2. Take any acyclic $N \in \mathbf{C}(A)$. By Proposition 10.1.3 we know that

$$\mathrm{Hom}_A(N, I) \cong \varprojlim_j \mathrm{Hom}_A(N, G_q(I))$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 2 the complexes $\mathrm{Hom}_A(N, G_q(I))$ are all acyclic. The exactness of the sequences (10.5.15) implies that the inverse system

$$\{\mathrm{Hom}_A(N, G_q(I))\}_{q \geq -1}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\mathrm{Hom}_A(N, I)$ is acyclic. \square

Theorem 10.5.16. *Let A be a DG ring. Any DG A -module M admits a quasi-isomorphism $\rho : M \rightarrow I$ in $\mathbf{C}_{\mathrm{str}}(A)$ to a semi-cofree DG A -module I .*

We shall need three lemmas before the proof of the theorem.

Lemma 10.5.17. *Let W be a cofree DG \mathbb{K} -module, let M be a DG A -module, and let $\chi : H(M) \rightarrow H(W)$ be a homomorphism in $\mathbf{G}^0(\mathbb{K})$. Then there is a homomorphism $\phi : M \rightarrow I_W$ in $\mathbf{C}_{\mathrm{str}}(A)$, such that the diagram*

$$\begin{array}{ccccc} H(M) & \xrightarrow{H(\phi)} & H(I_W) & \xrightarrow{H(\theta_W)} & H(W) \\ & \searrow \chi & & \nearrow & \\ & & & & \end{array}$$

in $\mathbf{G}^0(\mathbb{K})$ is commutative.

Proof. We can assume that

$$W = \prod_{p \in \mathbb{Z}} \prod_{s \in S^p} T^{-p}(\mathbb{K}^*)$$

for some graded set $S = \coprod_{p \in \mathbb{Z}} S^p$. Then

$$I_W = \prod_{p \in \mathbb{Z}} \prod_{s \in S^p} T^{-p}(A^*),$$

where $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}^*)$ as before. The trace θ_W is a product of translations of the trace $\theta : A^* \rightarrow \mathbb{K}^*$. The homomorphism $\chi : H(M) \rightarrow H(W)$ is a product of \mathbb{K} -linear homomorphisms $\chi_s : H^p(M) \rightarrow \mathbb{K}^*$. We see that it suffices to find, for each p and each $s \in S^p$, a homomorphism $\phi_s : M \rightarrow T^{-p}(A^*)$ in $\mathbf{C}_{\mathrm{str}}(A)$, such that $\theta \circ H^p(\phi_s) = \chi_s$.

Now we consider the simplified situation: $\chi : H^p(M) \rightarrow \mathbb{K}^*$ is a \mathbb{K} -linear homomorphism, and we are looking for a homomorphism $\phi : M \rightarrow T^{-p}(A^*)$ in $\mathbf{C}_{\mathrm{str}}(A)$ such that $\theta \circ H^p(\phi) = \chi$.

For any integer p let

$$Y^p(M) := \mathrm{Coker}(d_M^{p-1} : M^{p-1} \rightarrow M^p).$$

See Remark 7.3.10. In each degree p there are canonical exact sequences of \mathbb{K} -modules

$$(10.5.18) \quad 0 \rightarrow B^p(M) \rightarrow M^p \rightarrow Y^p(M) \rightarrow 0$$

and

$$(10.5.19) \quad 0 \rightarrow H^p(M) \rightarrow Y^p(M) \xrightarrow{d} B^{p+1}(M) \rightarrow 0.$$

Because \mathbb{K}^* is injective in $\mathbf{M}(\mathbb{K})$, we can extend $\chi : H^p(M) \rightarrow \mathbb{K}^*$ to a homomorphism $\chi : Y^p(M) \rightarrow \mathbb{K}^*$ in $\mathbf{M}(\mathbb{K})$, relative to the embedding $H^p(M) \hookrightarrow Y^p(M)$ in (10.5.19). Next we compose with the surjection $M^p \rightarrow Y^p(M)$ in (10.5.18) to obtain a \mathbb{K} -linear homomorphism $\chi : M^p \rightarrow \mathbb{K}^*$. Note that $\chi \circ d_M^{p-1} = 0$ by the exact sequence (10.5.18).

We now view χ as a degree $-p$ homomorphism $\chi : M \rightarrow \mathbb{K}^*$ in $\mathbf{G}(\mathbb{K})$ that sends all other components of M to zero. As an element of the DG \mathbb{K} -module $\text{Hom}_{\mathbb{K}}(M, \mathbb{K}^*)$, χ is a degree $-p$ cocycle. By adjunction (Lemma 10.5.10) we get an element $\psi \in \text{Hom}_A(M, A^*)$, and it is a cocycle of degree $-p$. Then

$$\phi := t^{-p} \circ \psi : M \rightarrow T^{-p}(A^*)$$

is a homomorphism in $\mathbf{C}_{\text{str}}(A)$ with the desired property. \square

Lemma 10.5.20. *In the situation of Lemma 10.5.17, let $N := \text{Cone}(T^{-1}(\phi))$, the standard cone on the homomorphism*

$$T^{-1}(\phi) : T^{-1}(M) \rightarrow T^{-1}(I_W)$$

in $\mathbf{C}_{\text{str}}(A)$. Consider the canonical exact sequence

$$(10.5.21) \quad 0 \rightarrow T^{-1}(I_W) \rightarrow N \xrightarrow{\pi} M \rightarrow 0$$

in $\mathbf{C}_{\text{str}}(A)$, shown in formula (3.9.4). Then the composed homomorphism

$$\chi \circ H(\pi) : H(N) \rightarrow H(W)$$

in $\mathbf{G}^0(\mathbb{K})$ is zero.

Proof. Passing to the long exact sequence in cohomology of (10.5.21), and then applying $\text{Hom}_{\mathbb{K}}(-, W^p)$, we obtain this long exact sequence:

$$\cdots \rightarrow \text{Hom}_{\mathbb{K}}(H^p(I_W), W^p) \rightarrow \text{Hom}_{\mathbb{K}}(H^p(M), W^p) \rightarrow \text{Hom}_{\mathbb{K}}(H^p(N), W^p) \rightarrow \cdots$$

in $\mathbf{M}(\mathbb{K})$. The homomorphism $\chi^p : H^p(M) \rightarrow W^p$ in the middle term comes from $H^p(\theta_W) : H^p(I_W) \rightarrow W^p$ in the left term. Therefore its image $\chi^p \circ H^p(\pi) : H^p(N) \rightarrow W^p$ in the right term is zero. \square

Lemma 10.5.22. *In the situation of Lemma 10.5.20, suppose $\rho : L \rightarrow M$ is a homomorphism in $\mathbf{C}_{\text{str}}(A)$ such that $H(\theta_W) \circ H(\phi) \circ H(\rho) = 0$. Then there exists a homomorphism $\sigma : L \rightarrow N$ in $\mathbf{C}_{\text{str}}(A)$ such that $\pi \circ \sigma = \rho$.*

See the next commutative diagrams, in $\mathbf{C}_{\text{str}}(A)$ and $\mathbf{G}^0(\mathbb{K})$ respectively.

$$\begin{array}{ccccccc} & & N & & & & \\ & \nearrow & \downarrow \pi & \searrow & & & \\ L & \xrightarrow{\rho} & M & \xrightarrow{\phi} & I_W & \xrightarrow{\theta_W} & W \\ & \searrow & \downarrow \xi & \nearrow & & & \end{array}$$

$$\begin{array}{ccccccc}
& & H(N) & & & & \\
& \nearrow^{H(\sigma)} & \downarrow H(\pi) & & & & \\
H(L) & \xrightarrow{H(\rho)} & H(M) & \xrightarrow{H(\phi)} & H(I_W) & \xrightarrow{H(\theta_W)} & W \\
& \searrow & & & & & \\
& & H(\xi)=0 & & & &
\end{array}$$

Proof. It will be convenient to express $N = \text{Cone}(T^{-1}(\phi))$ in terms of matrices. We will use the equality $T(T^{-1}(M)) = M$ to write the graded A^{\natural} -module N^{\natural} as a column:

$$(10.5.23) \quad N^{\natural} = \begin{bmatrix} T^{-1}(I_W)^{\natural} \\ T(T^{-1}(M))^{\natural} \end{bmatrix} = \begin{bmatrix} T^{-1}(I_W)^{\natural} \\ M^{\natural} \end{bmatrix}.$$

A small calculation, using Definition 3.8.5 and Proposition 3.8.10(1), shows that

$$T^{-1}(\phi) = t_{T^{-1}(I_W)}^{-1} \circ \phi \circ t_{T^{-1}(M)}.$$

Note that

$$t_{T^{-1}(I_W)} : T^{-1}(I_W) \rightarrow T(T^{-1}(I_W)) = I_W$$

is an invertible degree -1 homomorphism, and its inverse

$$t_{T^{-1}(I_W)}^{-1} : I_W \rightarrow T^{-1}(I_W)$$

has degree 1. So the differential of N is

$$(10.5.24) \quad d_N = \begin{bmatrix} d_{T^{-1}(I_W)} & T^{-1}(\phi) \circ t_{T^{-1}(M)}^{-1} \\ 0 & d_M \end{bmatrix} = \begin{bmatrix} d_{T^{-1}(I_W)} & t_{T^{-1}(I_W)}^{-1} \circ \phi \\ 0 & d_M \end{bmatrix}.$$

Define $\xi := \theta_W \circ \phi \circ \rho$. This is a homomorphism $\xi : L \rightarrow W$ in $\mathbf{C}_{\text{str}}(\mathbb{K})$, and by assumption $H(\xi) = 0$. According to Lemma 10.5.5, ξ is a coboundary in the DG module $\text{Hom}_{\mathbb{K}}(L, W)$. So there is some $\omega \in \text{Hom}_{\mathbb{K}}(L, W)^{-1}$ such that $\xi = d(\omega) = \omega \circ d_L$. Let $\alpha : L \rightarrow I_W$ be the unique A -linear homomorphism of degree -1 such that $\theta_W \circ \alpha = \omega$; see Lemma 10.5.10. Define the homomorphism $\sigma : L^{\natural} \rightarrow N^{\natural}$ in $\mathbf{G}^0(A^{\natural})$ to be the column

$$\sigma := \begin{bmatrix} t^{-1} \circ \alpha \\ \rho \end{bmatrix},$$

where from here to the end of the proof we write $t := t_{T^{-1}(I_W)}$. It is clear that $\pi \circ \sigma = \rho$.

It remains to prove that σ is strict, namely that $\sigma \circ d_L = d_N \circ \sigma$. Let us write out these homomorphisms as matrices. We have

$$\sigma \circ d_L = \begin{bmatrix} t^{-1} \circ \alpha \circ d_L \\ \rho \circ d_L \end{bmatrix}$$

and

$$d_N \circ \sigma = \begin{bmatrix} d_{T^{-1}(I_W)} & t^{-1} \circ \phi \\ 0 & d_M \end{bmatrix} \circ \begin{bmatrix} t^{-1} \circ \alpha \\ \rho \end{bmatrix} = \begin{bmatrix} d_{T^{-1}(I_W)} \circ t^{-1} \circ \alpha + t^{-1} \circ \phi \circ \rho \\ d_M \circ \rho \end{bmatrix}.$$

Since ρ is strict, there is equality $\rho \circ d_L = d_M \circ \rho$. We need to verify that

$$t^{-1} \circ \alpha \circ d_L = d_{T^{-1}(I_W)} \circ t^{-1} \circ \alpha + t^{-1} \circ \phi \circ \rho$$

as A -linear homomorphisms $L \rightarrow T^{-1}(I_W)$. We are allowed to postcompose with t ; so now we have to verify that

$$\alpha \circ d_L = t \circ d_{T^{-1}(I_W)} \circ t^{-1} \circ \alpha + \phi \circ \rho$$

as A -linear homomorphisms $L \rightarrow I_W$. By adjunction (Lemma 10.5.10) it suffices to verify that they are equal as \mathbb{K} -linear homomorphisms after postcomposing with θ_W . But

$$\theta_W \circ t \circ d_{T^{-1}(I_W)} \circ t^{-1} \circ \alpha = -\theta_W \circ d_{I_W} \circ \alpha = -d_W \circ \theta_W \circ \alpha = 0;$$

and

$$\theta_W \circ \phi \circ \rho = \xi = \theta_W \circ \alpha \circ d_L.$$

□

Proof of Theorem 10.5.16. The proof is morally dual to that of Theorem 10.3.9, but the details are much more complicated. This is the strategy: we will construct an inverse system $\{G_q(I)\}_{q \geq -1}$ in $\mathbf{C}_{\text{str}}(A)$, and an inverse system of homomorphisms $G_q(\rho) : M \rightarrow G_q(I)$ in $\mathbf{C}_{\text{str}}(A)$. Then we will prove that the DG module $I := \lim_{\leftarrow q} G_q(I)$ is semi-cofree, and the homomorphism $\lim_{\leftarrow q} G_q(\rho) : M \rightarrow I$ is a quasi-isomorphism.

Step 1. In this step we handle $q = 0$. By Lemma 10.5.7 there is an injective homomorphism $\chi : H(M) \rightarrow W$ in $\mathbf{G}^0(\mathbb{K})$ for some cofree DG \mathbb{K} -module W . Next, by Lemma 10.5.17, there is a homomorphism $\phi : M \rightarrow I_W$ in $\mathbf{C}_{\text{str}}(A)$, such that $\chi = H(\theta_W) \circ H(\phi)$.

Define the cofree DG A -module $G_0(I) := I_W$ and the homomorphism

$$G_0(\rho) := \phi : M \rightarrow G_0(I).$$

Then the homomorphism

$$H(G_0(\rho)) : H(M) \rightarrow H(G_0(I))$$

is injective.

Step 2. In this step $q \geq 0$, and we are given the following: a DG A -module $G_q(I)$, a cofiltration $\{G_{q'}(I)\}_{-1 \leq q' \leq q}$ of $G_q(I)$, and an inverse system of homomorphisms $G_{q'}(\rho) : M \rightarrow G_{q'}(I)$ in $\mathbf{C}_{\text{str}}(A)$. These satisfy the following conditions: the homomorphisms

$$H(G_{q'}(\rho)) : H(M) \rightarrow H(G_{q'}(I))$$

in $\mathbf{G}^0(\mathbb{K})$ are injective for all $0 \leq q' \leq q$; $G_{-1}(I) = 0$; and the DG A -modules

$$\text{Ker}(G_{q'}(I) \rightarrow G_{q'-1}(I))$$

are cofree for all $0 \leq q' \leq q$.

Let N be the cokernel of $H(G_q(\rho))$. So there is a short exact sequence

$$(10.5.25) \quad 0 \rightarrow H(M) \xrightarrow{H(G_q(\rho))} H(G_q(I)) \xrightarrow{\alpha} N \rightarrow 0$$

in $\mathbf{G}^0(\mathbb{K})$. By Lemma 10.5.7 there is an injective homomorphism $\chi : N \rightarrow W$ in $\mathbf{G}^0(\mathbb{K})$ for some cofree DG \mathbb{K} -module W . Next, by Lemma 10.5.17, there is a homomorphism $\phi : G_q(I) \rightarrow I_W$ in $\mathbf{C}_{\text{str}}(A)$, such that

$$\chi \circ \alpha = H(\theta_W) \circ H(\phi)$$

as homomorphisms $H(G_q(I)) \rightarrow W$. Define the DG A -module

$$G_{q+1}(I) := \text{Cone}(T^{-1}(\phi)),$$

the standard cone on the strict homomorphism $T^{-1}(\phi)$. There is a canonical exact sequence

$$(10.5.26) \quad 0 \rightarrow T^{-1}(I_W) \rightarrow G_{q+1}(I) \xrightarrow{\mu_q} G_q(I) \rightarrow 0$$

in $\mathbf{C}_{\text{str}}(A)$. According to Lemma 10.5.20, the homomorphism

$$\chi \circ \alpha \circ H(\mu_q) : H(G_{q+1}(I)) \rightarrow W$$

in $\mathbf{G}^0(\mathbb{K})$ is zero. Since χ is an injective homomorphism, we conclude that the homomorphism $\alpha \circ H(\mu_q)$ in the commutative diagram below is zero.

$$(10.5.27) \quad \begin{array}{ccccc} H(G_{q+1}(I)) & & & & \\ \downarrow H(\mu_q) & \searrow 0 & & \searrow 0 & \\ H(G_q(I)) & \xrightarrow{\alpha} & N & \xrightarrow{\chi} & W \\ & \searrow H(\phi) & & \nearrow H(\theta_W) & \\ & & H(I_W) & & \end{array}$$

Step 3. We continue from step 2. We know from formula (10.5.25) and diagram (10.5.27) that

$$H(\theta_W) \circ H(\phi) \circ H(G_q(\rho)) = 0.$$

According to Lemma 10.5.22 there is a homomorphism

$$G_{q+1}(\rho) : M \rightarrow G_{q+1}(I)$$

in $\mathbf{C}_{\text{str}}(A)$ such that the diagram

$$(10.5.28) \quad \begin{array}{ccc} & G_{q+1}(I) & \\ G_{q+1}(\rho) \nearrow & \downarrow \mu_q & \\ M & \xrightarrow{G_q(\rho)} & G_q(I) \end{array}$$

in $\mathbf{C}_{\text{str}}(A)$ is commutative.

The next diagram, in $\mathbf{G}^0(\mathbb{K})$, is also commutative, and the bottom row is exact:

$$(10.5.29) \quad \begin{array}{ccccccc} & & H(G_{q+1}(I)) & & & & \\ & \nearrow H(G_{q+1}(\rho)) & \downarrow H(\mu_q) & \searrow 0 & & & \\ 0 & \longrightarrow & H(M) & \xrightarrow{H(G_q(\rho))} & H(G_q(I)) & \xrightarrow{\alpha} & N \longrightarrow 0 \end{array}$$

Let us define

$$L_q := \text{Im}(H(\mu_q)) \subseteq H(G_q(I)).$$

From diagram (10.5.29) we see that the homomorphism

$$(10.5.30) \quad H(G_q(\rho)) : H(M) \rightarrow L_q$$

in $\mathbf{G}^0(\mathbb{K})$ is bijective. This implies that

$$\mathrm{H}(G_{q+1}(\rho)) : \mathrm{H}(M) \rightarrow \mathrm{H}(G_{q+1}(I))$$

in an injective homomorphism, a fact that is needed to keep the induction going.

Step 4. Proceeding with steps 2 and 3 inductively, we obtain an inverse system $\{G_q(I)\}_{q \geq -1}$ of objects in $\mathbf{C}_{\mathrm{str}}(A)$, and an inverse system $G_q(\rho) : M \rightarrow G_q(I)$ of homomorphisms in $\mathbf{C}_{\mathrm{str}}(A)$. The DG module $I := \lim_{\leftarrow q} G_q(I)$ comes equipped with the semi-cofree cofiltration $\{G_q(I)\}_{q \geq -1}$, and thus it is semi-cofree.

It remains to prove that the homomorphism

$$\rho := \lim_{\leftarrow q} G_q(\rho) : M \rightarrow I$$

is a quasi-isomorphism. From formula (10.5.30) we know that $\mathrm{H}(M) \rightarrow \lim_{\leftarrow q} L_q$ is bijective. The inverse systems $\{L_q\}_{q \geq 0}$ and $\{\mathrm{H}(G_q(I))\}_{q \geq 0}$ are sandwiched, so by Proposition 10.1.1(2) the limit of the second inverse system exists, and the canonical homomorphism

$$\lim_{\leftarrow q} L_q \rightarrow \lim_{\leftarrow q} \mathrm{H}(G_q(I))$$

is bijective.

Finally, the inverse systems $\{G_q(I)\}_{q \geq 0}$ and $\{\mathrm{H}(G_q(I))\}_{q \geq 0}$ satisfy the ML condition: the first has surjective transitions, and the images of the transitions $\mathrm{H}(G_{q'}(I)) \rightarrow \mathrm{H}(G_q(I))$ are stationary for $q' \geq q + 1$. Therefore the homomorphism

$$\mathrm{H}(I) \rightarrow \lim_{\leftarrow q} \mathrm{H}(G_q(I))$$

is bijective. Putting these facts together, we deduce that ρ is a quasi-isomorphism. \square

Corollary 10.5.31. *Let A be any DG ring. The category $\mathbf{C}(A)$ has enough K -injectives.*

Proof. Combine Theorems 10.5.13 and 10.5.16. \square

Corollary 10.5.32. *Assume A is a nonpositive DG ring (Definition 7.3.11). For any $M \in \mathbf{C}(A)$ there is a K -injective resolution $M \rightarrow I$ with $\inf(I) = \inf(\mathrm{H}(M))$.*

Proof. For any cofree DG \mathbb{K} -module W , the cofree DG A -module I_W has $\inf(I_W) = \inf(W)$. (Assuming that A is nonzero.) Looking at steps 1 and 2 of the proof of Theorem 10.5.16, we see that the DG modules $G_q(I)$ can be chosen such that $\inf(G_q(I)) = \inf(\mathrm{H}(M))$. \square

Remark 10.5.33. The proof of Theorem 10.5.16 is quite long and complicated. It would be nice to have a quicker proof.

In Keller's paper [Ke, Section 3.2] there is a slick proof of an even stronger result than Theorem 10.5.16 – but we were unable to understand the details!

Second Part [to be written later]**11. DERIVED BIFUNCTORS [LATER]**

- 11.1. **T-Additive Bifunctors [later].**
- 11.2. **Right Derived Bifunctors [later].**
- 11.3. **Left Derived Bifunctors [later].**
- 11.4. **The Right Derived Hom Bifunctor [later].**
- 11.5. **The Left Derived Tensor Bifunctor [later].**
- 11.6. **Adjunctions [later].**

12. DERIVED CATEGORIES IN GEOMETRY [LATER]

- 12.1. **Recalling Facts on Ringed Spaces.**
- 12.2. **K-Flat Resolutions in $\mathbf{C}(\mathcal{A})$ [later].**
- 12.3. **K-Injective Resolutions in $\mathbf{C}(\mathcal{A})$ [later].**
- 12.4. **K-Flasque Resolutions in $\mathbf{C}(\mathcal{A})$ [later].**
- 12.5. **Standard Derived Functors in Geometry [later].**
- 12.6. **Survey: Poincaré-Verdier Duality [later, optional].**
- 12.7. **Survey: Applications to Birational Geometry [later, optional].**

13. DERIVED CATEGORIES IN COMMUTATIVE ALGEBRA [LATER]

- 13.1. **Dualizing Complexes over Commutative Rings [later].**
- 13.2. **Local Duality [later].**
- 13.3. **Rigid Dualizing Complexes [later].**
- 13.4. **Survey: MGM Equivalence for Adic Rings [later, optional].**

14. RESIDUES AND DUALITY IN ALGEBRAIC GEOMETRY [LATER]

- 14.1. **Dualizing Complexes on Schemes [later].**
- 14.2. **Rigid Residue Complexes on Schemes [later].**
- 14.3. **Perverse Coherent Sheaves on Schemes [later, optional].**
- 14.4. **Grothendieck Duality for Proper Maps [later].**
- 14.5. **Survey of Related Material [later, optional].**

15. DERIVED CATEGORIES IN NONCOMMUTATIVE ALGEBRA [LATER]

15.1. **Noncommutative Dualizing Complexes** [later].

15.2. **Noncommutative Tilting Complexes** [later].

15.3. **The Noncommutative Derived Picard Group of a Ring** [later].

15.4. **Derived Morita Theory** [later].

Back to First Part [written]

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